Math 421-521

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Due February 8

Take home exam No. 1

I. Find:

1. $$\frac{1+2i}{3-4i}+\frac{2-i}{5i}=-\frac{2}{5}$$
2. $$\frac{5}{(1-i)(2-i)(3-i)}=\frac{i}{2}$$
3. $$(1-i)^4=-4$$
4. $$\frac{1}{(1+i)^2}=-\frac{i}{2}$$
5. $$\left(\frac{1}{2-3i}\right)\left(\frac{1}{1+i}\right)=\frac{5+i}{26}$$

II. Present in polar form:

6. $$-\frac{2}{1+i\sqrt{3}}=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=e^{\frac{2\pi i}{3}}$$
7. $$\frac{i}{-2-2i}=\frac{1}{2\sqrt{2}}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)=\frac{1}{2\sqrt{2}}\left(\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}\right)$$
8. $$\left(\sqrt{3}-i\right)^6=64\left(\cos\pi+i\sin\pi\right)=-64$$

III. Present in rectangular coordinates:

9.

$$3e^{(\pi i/3)} = \frac{3}{2}\left(1 + i\sqrt{3}\right)$$

10.

$$\sqrt{2}\, e^{-(\pi i/6)} = \frac{1}{\sqrt{2}}\left(\sqrt{3} + i\right)$$

11.

$$e^{(\pi i/4)} = \frac{1}{\sqrt{2}}\,(1 + i)$$

IV. Find all roots in rectangular coordinates, exhibit them geometrically, and point out, which one is the principal root:

12.

$$(2i)^{1/2} = \pm(1 + i), \qquad 1 + i\text{— principal}$$

13.

$$\left(1 - \sqrt{3}\, i\right)^{1/2} = \pm\frac{\sqrt{3} - i}{\sqrt{2}}, \qquad \frac{\sqrt{3} - i}{\sqrt{2}}\text{— principal}$$

14.

$$(-1)^{1/3} = e^{\pi i/3}, -1, e^{-\pi i/3}, \qquad e^{\pi i/3}\text{— principal}$$

15.

$$(-16)^{1/4} = \pm\sqrt{2}(1 + i), \pm\sqrt{2}(1 - i), \qquad \sqrt{2}(1 + i)\text{— principal}$$

16.

$$8^{1/6} = \pm\sqrt{2}, \pm\frac{1 + i\sqrt{3}}{\sqrt{2}}, \pm\frac{-1 + i\sqrt{3}}{\sqrt{2}}, \qquad \sqrt{2}\text{— principal}$$

V. Sketch the following sets and determine which are domain:

17.

$$|z - 2 + i| \leq 1$$

18.

$$|2z + 3| > 4$$

19.

$$Im\ z > 1$$

20.

$Im\ z = 1$

21.

$0 \leq arg\ z \leq \pi/4$

22.

$|z - 4| \geq Re z$

VI. Find $f'(z)$, if:

23.

$f(z) = (1 - 4z^2)^3 \qquad f'(z) = -24z(1 - 4z^2)^2$

24.

$f(x) = \dfrac{z-1}{2z+1} \quad \left(z \neq -\dfrac{1}{2}\right) \qquad f'(z) = \dfrac{3}{(2z+1)^2}$

25.

$f(z) = \dfrac{(1+z^2)^4}{z^3} \quad (z \neq 0) \qquad f'(z) = \dfrac{5z^2-3}{4}(1+z^2)^3$

VII. Check the Cauchy-Riemann condition for following functions

26.

$f(z) = 3x + y + i(3y - x) \qquad$ satisfied

27.

$f(z) = \sin x \cosh y + i \cos x \sinh y \qquad$ satisfied

28.

$f(z) = e^{-y}e^{ix} = e^{-y}(\cos x + i \sin x) \qquad$ satisfied

29.

$f(z) = xy + iy \qquad$ not satisfied

30.

$f(z) = e^{y}e^{ix} = e^{y}(\cos x + i \sin x) \qquad$ not satisfied

VIII. Show that $u(x, y)$ is harmonic in some domain and find the harmonic conjugate $v(x, y)$ if

31.

$u = 2x(1 - y) \qquad v = 2y + x^2 - y^2$

32.

$u = 2x - x^3 + 3xy^2 \qquad v = 2y - 3x^2y + y^3$

33.

$u = \sinh x \sin y \qquad v = -\cosh x \cos y$

34.

$u = \dfrac{y}{x^2 + y^2} \qquad v = -\dfrac{x}{x^2 + y^2}$

XI. Find all roots of the equations

35.

$e^z = -2 \qquad z = \ln 2 + \pi i(1 + 2n)$

36.

$e^z = 1 + \sqrt{3}i \qquad z = \ln 2 + \pi i(\dfrac{1}{3} + 2n)$

37.

$\exp(2z - 1) = 1 \qquad z = \dfrac{1}{2} + \pi n i$

38.

$\cosh z = \dfrac{1}{2} \qquad z = (2n \pm \dfrac{1}{3})\pi i$

39.

$\sinh z = i \qquad z = \dfrac{i\pi}{2} + 2i\pi n$

40.

$\sin z = \cosh 4 \qquad z_{1,2} = \pm\dfrac{\pi}{2} + 2\pi n - i \ln(\sqrt{\cosh^2 4 + 1} \pm \cosh 4)$

41.

$\log z = \dfrac{\pi}{2} i \qquad z = e^{\pi i/2}$

X. Find all the values of

42.

$$(1+i)^i = e^{-\pi/4+2\pi n+\frac{i}{2}\ln 2}$$

43.

$$(-1)^{1/\pi} = e^{i(2n+1)}$$

44.

$$i^i = e^{-\pi/2+2\pi n}$$

45.

$$(1-i)^{4i} = e^{-\pi(1+8n)+2i\ln 2}$$

46.

$$\tan^{-1}(2i) = (n+\frac{1}{2})\pi$$

47.

$$\tan^{-1}(1+i) = \frac{1}{4}\ln 5 + i\left(\pi n - \frac{1}{2}\arctan 2\right)$$

48.

$$\tanh^{-1} 0 = n\pi i$$

49.

$$\cosh^{-1}(-1) = \pi i(1+2n)$$