Math 421-521

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Due March 22

Homework No. 2

1. Obtain the Maclaurin series representation

$$z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \qquad (|z| < \infty)$$

2. Find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9}\left[\frac{1}{1+(z^4/9)}\right]$$

3. Write the Maclaurin series representation of the function $f(z) = \sin(z^2)$, and point out how it follows that

$$f^{(4n)}(0) = 0 \quad and \quad f^{(2n=1)}(0) = 0 \quad (n = 0, 1, 2, ...)$$

4. Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)}$$

and specify the regions in which those expansions are valid.

5. Write the two Laurent series in powers of z that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains, and specify those domains.

6. Write the principal part of the function at its isolated singular point.

$$z\,\exp\left(\frac{1}{z}\right)$$

7. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

$$(a)\ \frac{1-\cosh z}{z^3}; \quad (b)\frac{1-\exp(2z)}{z^4}$$

8. Find the residue at $z = 0$ of the function

$$(a)\ \frac{1}{z+z^2}; \quad (b)\ z\cos\left(\frac{1}{z}\right); \quad (c)\ \frac{\cot z}{z^4}$$

9. Find the value of the integral

$$\int_c \frac{3z^3+2}{(z-1)(z^2+9)}dz,$$

taken counterclockwise around the circle

(a) $|z-2|=2$; $\quad(b)$ $|z|=4$

10. Find the value of the integral

$$\int_c \frac{dz}{z^3(z+4)}$$

taken counterclockwise around the circle

(a) $|z|=2$; $\quad(b)$ $|z+2|=3$

11. Let C be the circle $|z|=2$, described in the positive way, and evaluate the integral

$$\int_c \frac{dz}{\sinh 2z}$$

12. Use expression

$$\int_c f(z)dz = 2\pi i Res_{z=0}\, \frac{1}{z^2} f\left(\frac{1}{z}\right),$$

involving a single residue, to evaluate the integral of $f(z)$ around the positively oriented circle $|z|=3$ when $f(z)$ is

$$\frac{(3z+2)^2}{z(z-1)(2z+5)}$$

Evaluate the integrals:

13.

$$\int_0^\infty \frac{dx}{x^2+1}$$

14.

$$\int_0^\infty \frac{x^2dx}{(x^2+1)(x^2+4)}$$

15.

$$\int_0^\infty \frac{dx}{x^4+1}$$

16.

$$\int_0^\infty \frac{\cos ax}{x^2+1}dx$$

17.

$$\int_0^\infty \frac{x \sin 2x}{x^2+3} dx$$

18.

$$\int_{-\infty}^\infty \frac{x \sin ax}{x^4+4} dx$$

19.

$$\int_{-\infty}^\infty \frac{x^2 dx}{(x^2+1)^2}$$

20.

$$\int_{-\infty}^\infty \frac{x \sin x dx}{(x^2+1)(x^2+4)}$$

21.

$$\int_{-\infty}^\infty \frac{\cos x dx}{(x+a)^2+b^2} \quad (b>0)$$

22.

$$\int_0^\infty \frac{\ln x}{x^2+1} dx$$

23.

$$\int_0^\infty \frac{\ln x}{(x^2+1)^2} dx$$

24.

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$$

25.

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}$$

26.

$$\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta}$$

27.

$$\int_0^{\pi} \frac{\cos 2\theta \, d\theta}{1-2a\cos\theta+a^2}$$

28.

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)}$$

29.

$$\int_0^\infty \frac{x^a}{(x^2+1)^2} dx$$

30. Determine the number of zeros of the polynomial

$$(a)\ z^6 - 5z^4 + z^3 - 2z; \quad (b)\ 2z^4 - 2z^3 + 2z^2 - 2z + 9$$

inside the circle $|z| = 1$.

31. Determine the number of roots of the equation

$$2z^5 - 6z^2 + z + 1 = 0$$

in the region $1 \leq |z| < 2$.