Math 422-522

Lecture 8 February 5 2019

① We will study linear ODE of second order

$$y'' + p(x)y' + q(x)y = 0 \qquad (1)$$

Equation (1) has two linearly independent solutions

$y_1(x)$, $y_2(x)$

Let

$$I = \{y_1, y_2\} = y_1' y_2 - y_2' y_1 \quad \text{— the Wronskian}$$

$$\varphi = \int_{x_0}^{x} p(x)\,dx \qquad p = \varphi'$$

From (1) we obtain

$$\frac{dI}{dx} + pI = 0$$

$$I = c\,e^{-\varphi} \qquad c \text{ — a constant}$$

Two auxilary but important facts

(A) If $y_1(x)$ is known $y_2(x)$ can be found

Indeed

$$y_1' y_2 - y_2' y_1 = c\, e^{-\varphi}$$

Let $y_2 = f y_1$. Then

$$-f' y_1^2 = c\, e^{-\varphi} \qquad f' = -\frac{c}{y_1^2} e^{-\varphi}$$

(B) Let us perform the transform

$$y = z\, e^{-\frac{\varphi}{2}}$$

This transform kills the first derivative case

z satisfies to equation

$$z'' + \tilde{q}(x) z = 0$$

$$\tilde{q}(x) = q(x) - \frac{p^2}{4} - \frac{p'}{2}$$

In the special case

$$p = -\frac{2}{x - x_0} \qquad \tilde{q} = q$$

Euler-Cauchy equations

$$a_2 x^2 y'' + a_1 x y' + a_0 y = 0$$

we assume $a_2 \neq 0$

Then

$$p = \frac{a_1}{a_2} \frac{1}{x} \qquad q = \frac{a_0}{a_2} \frac{1}{x^2}$$

One can look for solutions in the form

$$y = x^m$$

m satisfies the "indicial equation"

$$a_2 m^2 + a_1 m + a_0 = 0 \qquad (2)$$

If $\Delta = a_1^2 - 4a_0a_2 \neq 0$, this equation has two powerlike solutions

$$y_1 = x^{m_1} \qquad y_2 = x^{m_2}$$

If $\Delta = 0$ and the root is double

$$y_1 = x^m \qquad y_2 = x^m \ln x$$

For instanse the equation

$$4x^2 y'' + y(x) = 0$$

has solutions

$$y_1 = x^{1/2} \qquad y_2 = x^{1/2} \ln x$$

Suppose that $a_0 = 0$

Then equation is

$$y'' + \frac{\lambda}{x} y' = 0$$

If $\lambda \neq 1$ it has solutions

$$y_1 = 1 \qquad y_2 = x^{1-\lambda}$$

If $\lambda = 1$ (such happens systematically)

$$y_1 = 1 \qquad y_2 = \ln x$$

$$y' + \lambda y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{1} n a_n x^{n-1} = \sum_{0}^{\infty} (n+1) a_{n+1} x^n$$

$$a_{n+1} = \cancel{\frac{a_n}{}} - \frac{a_n}{n+1}$$

$$a_1 = -\frac{\lambda a_0}{1} \qquad a_2 = \frac{\lambda a_1}{2} = \frac{\lambda^2 a_0}{1 \cdot 2}$$

$$a_n = (-1)^n \frac{\lambda^n}{n!} = e^{-\lambda x}$$

Now let

$$y'' + y(x) = 0$$

We know that

$$y_1 = \cos x$$

$$y_2 = \sin x$$

How to receive them by the Power Series Method?

Let us calculate y''

Apparently $\frac{d}{dx^2}(a_1 + a_2 x) = 0$

$$y''(x) = 2c_2 + 2\cdot 3\, c_3 x + 3\cdot 4\, c_4 x^2 + \dots$$

Hence

$2a_2 = -a_0$ $\qquad 2\cdot 3\, a_3 = -a_1$

$\cancel{n(n+1)c_{n+2} = -c_n}$ $\qquad (n+1)(n+2)\, a_{n+2} = -a_n$

$$\cancel{c_{n+2} = -\frac{c_n}{n(n+1)}}$$

$$a_{n+2} = -\frac{a_n}{(n+1)(n+2)}$$

$$a_2 = -\frac{a_0}{1\cdot 2} = -\frac{a_0}{2!}$$

$$a_4 = -\frac{a_2}{3\cdot 4} = \frac{a_0}{1\cdot 2\cdot 3\cdot 4} = \frac{a_0}{4!}$$

$$a_6 = -\frac{a_4}{5\cdot 6} = -\frac{a_0}{6!}$$

Finally

$$a_{2n} = (-1)^n \frac{a_0}{n!}$$

$$y = a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}\right) = a_0 \cos x$$

Then

$$a_3 = -\frac{a_1}{2\cdot 3} = -\frac{a_1}{3!}$$

$$a_5 = -\frac{a_3}{4\cdot 5} = \frac{a_1}{2\cdot 3\cdot 4\cdot 5} = \frac{a_1}{5!}$$

$$a_7 = -\frac{a_1}{7!}$$

$$a_{2n+1} = (-1)^{n} \frac{x}{(2n+1)!} x = (-1)^{n+1} \frac{1}{(2n+1)!}$$

$$y = a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right) = a_1 \sin x$$

$$y = a_0 \cos x + a_1 \sin x$$

Next example

$$y'' - 3xy' + 3y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$3a_0 + 3a_1 +$$

$$3xy' = \sum_{n=1}^{\infty} n a_n x^{n}$$

$$y'' = \sum_{n=0}^{\infty} n(n+1) a_{n+2} x^n$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2$$

$$xy' = a_1 x + 2a_2 x^2 + 3a_3 x^3$$

$$xy' = a_0 + \sum_{n=1} (n+1) a_n x^n$$

$$y + xy' = \sum_{n=0}^{\infty} (n+1) a_n x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+1)(n+2)\, \overset{-9}{a_{n+2}}\, x^n$$

$$(n+1)(n+2)\, a_{n+2} + 3(n+1)\, a_n = 0$$

$$a_{n+2} = - \frac{3 a_n}{n+2}$$

$$a_2 = -\frac{3}{2} \qquad a_4 = \frac{3}{2} \cdot \frac{3}{4} \qquad a_6 = -\frac{3 \cdot 3 \cdot 3}{2 \cdot 4 \cdot 6}$$

$$a_{2n} = (-1)^n \frac{3^n}{2^n} \frac{1}{n!}$$

$$a_3 = - \frac{3 a_1}{3}$$

$$a_5 = + \frac{3 a_3}{5} = \frac{9}{3 \cdot 5}$$

$$a_7 = - \frac{3 a_5}{7} = - \frac{3^3}{3 \cdot 5 \cdot 7}$$

$$a_{2n+1} = (-)^n \frac{u'}{(7n+1)!!}$$

$$y(x) = a_0 \sum (-1)^n \frac{3^n}{2^n} \frac{x^{2n}}{n!} + a_1 \sum \frac{(-1)^n 3^n}{(2n+1)!!} x^{2n+1}.$$

Next example

$$x^2 y'' + (3x+1) y = 0$$

$$y = \sum a_n x^n \qquad = a_0 + a_1 x + a_2$$

$$y'' = \cancel{\sum a_{n+2}} \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$x^2 y'' = \sum_{n=0} n(n-1) a_n x^n = \sum (n+1)(n+2) a_{n+2} x^n$$

$$\cancel{\sum n(n-1) a_n x^n}$$

$$x^2 y'' + y = \sum_{n=0}^{\infty} (n^2 - n + 1) a_n x^n$$

$$\cancel{3x} = a_0 + a_1 x + 3 a_2 x^2 + 7 a_3 x^3 + 13 a_4 x^4 +$$

$$+ 3 (a_0 x + a_1 x^2 + a_2 x^3 + \dots)$$