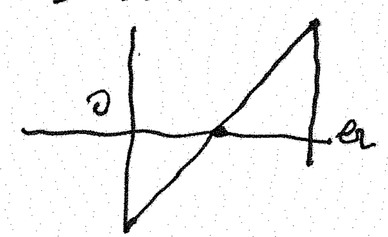
Math 422–S22

Midterm exam 2.

Practice test

1. Determine the cosine series of $f(x) = x - \frac{a}{2}$ defined on the intervale $[0, a]$

Solution. Looking at the plot of function we see that $\int_0^a f(x)\,dx = 0$



hence $a_0 = 0$

$$a_n = \frac{2}{\ell} \int_0^a \left(x - \frac{a}{2}\right) \cos \frac{\pi n}{a} x \, dx$$

$$\cos \frac{\pi n}{a} x = \frac{a}{\pi n} \frac{d}{dx} \sin \frac{\pi n}{a} x$$

Hence

$$a_n = \frac{2}{\ell} \left[\frac{a}{\pi n} \left(x - \frac{a}{2}\right) \sin \frac{\pi n}{a} x \Big|_0^a - \frac{a}{\pi n} \int_0^a \sin \frac{\pi n}{a} x \right]$$

Off-integral term is zero

$$a_n = \frac{2}{\ell}\left(\frac{a}{\pi n}\right)^2 \cos\frac{\pi n}{a}x \Big|_0^{\ell} = \frac{2}{\ell}\left(\frac{a}{\pi n}\right)^2\left[(-1)^n - 1\right]$$

$a_{2k} = 0$ ~~a_{2k+1}~~

$$a_{2k+1} = -\frac{4}{\ell}\left(\frac{a}{\pi(2k+1)}\right)^2$$

2. Solve the wave equation

$$u_{tt} = c\, u_{xx}$$

inside the intervale $0 < x < \ell$

with ~~Dirichlet~~ ~~Di~~ Dirichlet boundary conditions

$$u(0) = u(\ell) = 0$$

and initial data

$$u\big|_{t=0} = \sin\frac{\pi n}{\ell}x \qquad u_t\big|_{t=0} = 0$$

<u>Solution</u>

The solution can be found in ~~for~~ of a Fourier sine series. As far as only one

term is presented in the initial data,
The solution must have the form

$$u = \sin\frac{\pi n}{l}x\, T(t) \qquad \omega_n^2 = c^2\frac{\pi^2 n^2}{l^2}$$

$$T'' + \omega_n^2 T = 0 \qquad T\big|_{t=0} = 1 \qquad T'\big|_{t=0} = 0$$

hence $T(t) = \cos\omega_n t$

Finally

$$u = \sin\frac{\pi n}{l}x \cos\omega_n t$$

3. Find conditions when the polynomial

$$P = a_1 x^4 + a_2 y^4 + a_3 x^2 y^2$$

is a harmonic function. If these conditions are fulfilled, find the conjugated function Q

Solution

$$\Delta P = \frac{\partial^2}{\partial x^2}P + \frac{\partial^2 P}{\partial y^2} = (12a_1 + 2a_3)x^2 + (12a_2 + 2a_3)y^2$$

$\Delta P = 0$ if

Hence $a_1 = a_2 = a \qquad a_3 = -6a$

Thus

$P = a\,(x^4 + y^4 - 6x^2y^2)$ — This a homenous polynomial of forth order

The conjgate function also must be a homogenous polynomial of fourth order, different from P. The only choise is

$$Q = c\,(x^3y - xy^3)$$

The condition

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} \qquad \text{is satisied if} \qquad c = 4a$$

5. Find the Fourier Transform of the function

$$f(x) = \begin{cases} \cos x & |x| < \pi \\ 0 & |x| > \pi \end{cases}$$

$$\hat{F}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos x\, e^{-ikx}\,dx = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left[e^{(ik+i)x} + e^{(-ik-i)}\right]dx$$

$$\hat{F}(k) = \frac{1}{4\pi}\left[\frac{1}{i(1-k)} e^{i(1-k)x}\Big|_{-\pi}^{\pi} - \frac{1}{i(1+k)} e^{-i(1+k)x}\Big|_{-\pi}^{\pi}\right]$$

$$e^{i\pi} = e^{-i\pi} = -1$$

$$\hat{F} = -\frac{1}{4\pi i}\left[\frac{1}{1-k}\left(e^{-ik\pi} - e^{ik\pi}\right) - \frac{1}{1+k}\left(e^{-ik\pi} - e^{ik\pi}\right)\right] =$$

$$= \frac{1}{2\pi} \sin k\pi \left[\frac{1}{1-k} - \frac{1}{1+k}\right]$$

5. Find the Laplace transform of the function

$$f(t) = t^2 e^{-at}$$

By the use of shift and differention theorems one obtains immediately

$$\hat{F}(s) = \frac{2}{(s-a)^3}$$