Math 456-556

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Home Work No. 1. Solutions

1. Determine, which equations are linear:

   (a)
   $$u_t + x\, u_x + u_{xxx} = 0$$

   (b)
   $$u_{tt} - u_{xx} + u = 0 \quad (Klein - Gordon\ \ equation)$$

   (c)
   $$u_{tt} - u_{xx} + \sin u = 0 \quad (Sine - Gordon\ \ equation)$$

   (d)
   $$S_x^2 + S_y^2 = u(x, y) \quad (Eikonal\ \ equation)$$

   (e)
   $$\phi_t\, \phi_x + \phi\, \phi_{xt} + \phi_x^2 + \phi\, \phi_{xx} = 0$$

   (f)
   $$u_t + u\, u_x = u_{xx} \quad (Burgers\ \ equation)$$

   (g)
   $$u_{xy} = e^u \quad (Poincare\ \ equation)$$

   Equations (a), (b) are linear. All others - nonlinear. However, equations (e), (f) can be transformed to linear ones by a certain change of variables and equation (g) can be reduced to the ODE.

   (e). Suppose, $u = \phi\phi_x$. Equation (e) reads
   $$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

(f). In the Burgers equation substitution (Cole-Hopf anzatz)

$$u = -\frac{2}{\phi}\frac{\partial \phi}{\partial x}$$

leads to the linear equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

(g). Equation is essentially nonlinear but can be reduced to ODE by the certain change of variables. Let

$$u = \ln V_x \quad e^u = V_x$$

$$u_y = \frac{V_{xy}}{V_x} \quad \frac{\partial}{\partial x} u_y = V_x \quad u_y = V$$

$$V_{xy} = V V_x = \frac{1}{2}\frac{\partial}{\partial x} V^2$$

$$V_y = \frac{1}{2} V^2 + f(y)$$

Function $f(y)$ is an arbitrary function on y. It is a so-called Riccachi equation.

2. Find general solutions of the following linear first order equations:

   (a)
   $$x\, u_x + y\, u_y = 0$$

   (b)
   $$x\, u_x - y\, u_y = 0$$

   (c)
   $$x\, u_y + y\, u_x = 0$$

   (d)
   $$x\, u_y - y\, u_x = 0$$

(a).

$$\frac{dy}{dx} = \frac{y}{x} \quad \frac{dy}{y} = \frac{dx}{x}$$

$$y = cx \quad c = \frac{y}{x} \;\; is\;\; integral$$

$$u = f\left(\frac{y}{x}\right)$$

So, f is an arbitrary function.

(b).

$$\frac{dy}{dx} = -\frac{y}{x} \quad \frac{dy}{y} + \frac{dx}{x} = 0 \quad xy = c$$
$$u = f(xy)$$

(c).

$$xu_y + yu_x = 0 \quad \frac{dy}{dx} = \frac{x}{y} \quad x^2 - y^2 = c$$
$$f = f(x^2 - y^2)$$

(d).

$$xu_y - yu_x = 0 \quad \frac{dy}{dx} = -\frac{x}{y}$$
$$u = f(x^2 + y^2)$$

3. Solve the following equation:

$$u_x + u_y + u = e^{x+2y}$$

if $u(x, 0) = 0$.

$$u_x + u_y + u = e^{x+2y} \quad u = u_1 + u_2$$
$$u_1 = \frac{1}{4}e^{x+2y}$$

is a particular solution of the inhomogeneous equation.

$$u_{2x} + u_{2y} + u_2 = 0, \quad u_2 = e^{-x}\phi$$

$$\phi_x + \phi_y = 0, \quad \phi = f(x - y)$$
$$\frac{1}{4}e^{x+2y} + f(x - y)e^{-x} = u$$
$$u|_{y+0} = 0 \quad f(x) = -\frac{1}{4}e^{2x}$$

Finally:

$$u = \frac{1}{4}\left(e^{x+2y} - e^{x-2y}\right)$$

4. Solve problems 1,2,4 on page 31 of Strauss textbook.

(1.6.1 a)

$$u_{xx} - 4u_{xy} + u_{yy} + 2u_y + 4y = 0$$
$$a_{11} = 1 \quad a_{22} = 1 \quad a_{12} = 2$$
$$a_{11}a_{22} - a_{12} = -3 \ \ (hyperbolic)$$

(1.6.1 b)

$$9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$$
$$a_{11} = 9 \quad a_{22} = 1 \quad a_{12} = 3$$

This is a parabolic equation

$$u_x + \left(\frac{\partial}{\partial y} + 3\frac{\partial}{\partial x}\right)^2 = 0$$

(1.6.2)

$$(1 + x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$
$$a_{11} = 1 + x \quad a_{12} = xy \quad a_{22} = -y^2$$
$$a_{11}a_{22} - a_{12}^2 = -y^2(1 + x) - x^2y^2 = -y^2(1 + x + x^2) < 0$$

This equation is hyperbolic everywhere but the real axis $y = 0$.

(1.6.4)

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0$$
$$a_{11} = 1 \quad a_{22} = 4 \quad a_{12} = 2$$
$$\Delta = 1 \cdot 4 - 4 = 0$$

This equation can be written as follow

$$\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)^2 u = 0$$

$$\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)(f(y + 2x) + xg(y + 2x)) = g(y + 2x)$$

$$\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right) g(y + 2x) = 0$$

5. Show that

$$u = \sin nx \ \cos ny + 2\cos mx \ \sin my$$

satisfies the equation

$$u_{xx} - u_{yy} = 0$$

$$u_{xx} - u_{yy} = 0$$

$$u = \sin nx \ \cos ny + 2\cos mx \ \sin my$$

$$u_{xx} = -n^2 \sin nx \ \cos ny - 2m^2 \ \cos mx \ \sin my$$

$$u_{yy} = -n^2 \sin nx \ \cos ny - 2m^2 \ \cos mx \ \sin my$$

$$u_{xx} = u_{yy}$$