Math 456-556

Instructor: Vladimir Zakharov

Due February 11, 2009

Home Work No. 2.

1. The star is in the sphere of radius R. The temperature inside the star satisfies the heat transport equation with source

$$k\,\Delta T + f = 0.$$

Heat conductivity k and density of heat production f are constants. On the surface the heat flux from inner parts of the star transforms to radiation; it means that the following boundary condition is satisfied:

$$-k\frac{\partial T}{\partial r} = \sigma\,T^4|_{r=R}$$

Find the temperature in the center of the star.

2. Solve the wave equation

$$u_{tt} = c^2\,u_{xx} \qquad -\infty < x < \infty$$

with initial data

$$\phi = A\cos kx = u|_{t=0},$$

$$\psi = A\,\alpha\sin kx = u_t|_{t=0}.$$

Discuss specially the cases $\alpha = 0$ (standing wave) and $\alpha = \pm\,ck$ (propagating wave).

3. Solve the initial value problem

$$\phi = u|_{x=0} = 0 \quad \psi = u_t|_{t=o} = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

for the wave equation

$$u_{tt} = c^2 \, u_{xx} \qquad -\infty < x < \infty, \quad u \to 0 \;\; at \;\; x \to \pm\infty$$

Sketch the areas on the x, t-plane, where solutions have different analytic forms.

4. Find energy the conservation law for equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 (u_{xx} - \lambda^2 u_{xxxx}) \quad u \to 0, \; u_x \to 0, \;\; x \to \pm\infty$$

Prove the uniqueness theorem for the Cauchy problem

$$u|_{t=0} = \phi(x) \quad u_t|_{t=0} = \psi(x)$$

5. Prove the uniqueness theorem for the damping wave equation

$$u_{tt} = c^2 u_{xx} - 2\gamma(t, x) \, u_t, \quad \gamma(t, x) \geq 0$$

6. Solve the wave equation on the half-line $0 < x < \infty$, subject for Dirichlet conditions $u|_{x=0} = 0$ with initial data

$$u|_{t=0} = \phi(x) = \delta(x - x_0)$$

$$u_t|_{t=0} = \psi(x) = 0$$

Solve the same problem with Neumann boundary condition

$$u_x|_{x=0} = 0$$

7. Solve the radial wave equation

$$u_{tt} = \frac{c^2}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial u}{\partial r} \qquad u \to 0, \;\; r \to \infty$$

with the initial data

$$u = u_0 \, \delta(r - r_0), \quad u_t = c \, u_0 \, \delta'(r - r_0).$$

8. Solve the diffusion equation

$$u_t = k \, u_{xx} \qquad -\infty < x < \infty$$

$$u|_{t=0} = \phi(x) = e^{-x^2/L^2}$$

9. Prove the uniqueness theorem for the damping diffusion equation

$$u_t = k\, u_{xx} - \gamma(x,t)\, u \qquad \gamma \geq 0$$

10. Solve the diffusion equation

$$u_t = k\, u_{xx}$$

subject for inhomogeneous Dirichlet boundary conditions

$$u|_{x=0} = \sin \omega t \qquad u \to 0,\ \ x \to \infty$$

Find periodic in time solution

$$u = A(x)\, \sin \omega t + B(x)\, \cos \omega t$$

11. Exercise 8 on page 45 in the Strauss textbook.

12. Solve the diffusion equation

$$u_t = k\, u_{xx}$$

on the half-line $0 < x < \infty$ with initial data

$$u|_{t=0} = 0$$

and inhomogeneous Dirichlet boundary condition

$$u|_{x=0} = h(t).$$