On the Dressing Method

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Introduction

The purpose of this lecture is to present the development of the so-called "dressing method" in the theory of integrable systems. This method, allowing to find both new integrable systems and their exact solutions, is one of the most poweful tools in the nonlinear mathematical physics. It is clear now that a natural field for its applications is the theory of integrable nonlinear equations in 2+1 dimentions, in other words, the theory of integrable evolutional equations on (x,y) plane. We use here and further the words "integrable system" in a colloquial meaning – most of this systems are not integrable in a strict Louville's sense. Nevertheless they have infinite sets of motion integrals and exact solutions and they could be studied effectively. There are two different versions of the dressing method in 2+1 dimensions. The first one, based on Volterra-type linear integral equations, was introduced in 1974 in the article of A.B.Shabat and the author [1]. The second version, using $\bar{\partial}$-problem on the complex plane was developed by S.V.Manakov and the author in 1985 [2]. One of the purposes of my lecture is to show that both mentioned approaches are equivalent in some sense. This fact plays a key role in the further progress of the theory.

My second aim is to outline some possibilities for generalization of the dressing method. We shall see that this method can be expanded to cover broad new classes of integrable equations. It is a serious work to explore all the new capacities. We shall do just the first steps in this direction. I will display several new integrable equations mostly with space- and time-dependent coefficients and show also some new interesting solutions of previosly well known integrable equations.

1 The "old" dressing method.

Let us describe shortly the principal idea of article [1]. We start from the equation:

$$K(x,z) + F(x,z) + \int_x^\infty K(x,s)F(s,z)ds = 0 \qquad (1.1)$$

Here F is a known and K is unknown $N \times N$ matrix functions, depending also on some additional variables y_j $(j = 1, ..., n)$. The equation (1.1) could be rewritten in the symbolic form

$$K + F + K * F = 0 \tag{1.2}$$

The first step in the dressing method is finding ouf a pair of operators $D, \tilde{D}$ satisfying the equation

$$\tilde{D}K + DF + \tilde{D}K * F + K * DF = 0 \tag{1.3}$$

The operator D is *bare* while the operator $\tilde{D}$ is *dressed* one. An operator D is cold *dressible* if a corresponding $\tilde{D}$ could be found. The most simple dressible operators are differential. Let us put $N = 1$ (K and F are scalar functions) and choose the operator D in the form

$$DF = \alpha \frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial z^2} \tag{1.4}$$

Here α is an arbitrary constant. Applying then operator D to (1.1), we find after a simple calculation that (1.3) is valid if

$$\tilde{D}K = \alpha \frac{\partial K}{\partial y} + \frac{\partial^2 K}{\partial x^2} + U(x)K - \frac{\partial^2 K}{\partial z^2} \tag{1.5}$$

Here $U(x) = -2\frac{d}{dx}K(x,x)$. So the operator (1.4) is dressible. More general example of dressible operators was found in [1]. Let N be arbitrary and

$$L_0 = l_0(x, y, ...y_n)\frac{\partial^n}{\partial x^n} + ... \tag{1.6}$$

be a linear ordinary differential operator of order n. All its coefficients $l_i (i = 0, 1, ...n)$ are $N \times N$ matrix functions. Let us consider the operator

$$DF = \alpha \frac{\partial F}{\partial y} + L_0 F - F L_0^+ \tag{1.7}$$

Here L_0^+ is a coadjoned operator acting from right side on the variable z. The central result of the article [1] is formulated as a following: *the operator (1.7) is dressible.* The dressed operator $\tilde{D}$ has a form

$$\tilde{D}K = \alpha \frac{\partial K}{\partial y} + LK - KL_0^+ \tag{1.8}$$

Here $L = L_0 + \tilde{L}$ and

$$\tilde{L} = \tilde{l}_0 \frac{\partial^{n-1}}{\partial x^{n-1}} + ... \tag{1.9}$$

is a linear differential operator of the order $n - 1$. Its coefficients $\tilde{l}_i$ depend upon the function $K(x,z,y_i)$. Let us consider the set of functions

$$K_i(x,y) = (\frac{\partial}{\partial x} - \frac{\partial}{\partial z})^i K(x,y,z)|_{z=x} \qquad (i = 0,1,...,n-1)$$

Here coefficients $\tilde{\ell}_i$ are polinomial on $K_j(0 \leq j \leq i)$ and on their x-derivatives. In the previous example we had

$$L_0 = \frac{\partial^2}{\partial x^2} \qquad L = \frac{\partial^2}{\partial x^2} + U(x) \qquad U = -2\frac{d}{dx}K_0$$

In this case $\tilde{\ell}_0 = 0$, $\tilde{\ell}_1 = U$. To construct an integrable system we should have two *commuting* dressible operators

$$\begin{aligned} D_1 F &= \alpha_1 \frac{\partial F}{\partial y_1} + L_0^1 F - F L_0^{1+} \\ D_2 F &= \alpha_2 \frac{\partial F}{\partial y_2} + L_0^2 F - F L_0^{2+} \end{aligned} \tag{1.10}$$

The condition of commutativity $[D_1 D_2] = 0$ gives

$$\alpha_1 \frac{\partial L_0^2}{\partial y_1} - \alpha_2 \frac{\partial L_0^1}{\partial y_2} + [L_0^1, L_0^2] = 0 \tag{1.11}$$

Assume now that

$$D_1 F = 0 \qquad D_2 F = 0 \tag{1.12}$$

In accordance with (1.11) these equations are compatible. Supposing that the equation (1.2) could be resolved uniquely, we have

$$\tilde{D}_1 K = 0 \qquad \tilde{D}_2 K = 0 \tag{1.13}$$

and hence

$$\alpha_1 \frac{\partial L^2}{\partial y_1} - \alpha_2 \frac{\partial L^1}{\partial y_2} + [L^1, L^2] = 0 \tag{1.14}$$

From (1.14) and (1.11) we find

$$\alpha_1 \frac{\partial \tilde{L}^2}{\partial y_1} - \alpha_2 \frac{\partial \tilde{L}^1}{\partial y_2} + [L_0^1, \tilde{L}^2] + [\tilde{L}^1, L_0^2] + [\tilde{L}^1, \tilde{L}^2] = 0 \tag{1.15}$$

This is a system of nonlinear equations on the coefficients of operators $\tilde{L}^{1,2}$. This system is imposed on the finite set of functions K_i in fact. After solving the equation (1.1) we can get $K(x,z,y,...y_n)$ and some definite solutions of this system. Assuming that

$$L_0^2 = \frac{\partial^3}{\partial x^3} \tag{1.16}$$

we have after some computations the expression

$$L^2 = \frac{\partial^3}{\partial x^3} + 6U\frac{\partial}{\partial x} + W \tag{1.17}$$

Here

$$W = 12\frac{\partial^2}{\partial x^2}K_0 + 12\frac{\partial}{\partial x}K_1 - GUK_0 = 3U_x + 6\alpha\frac{\partial}{\partial y}K_0$$

Considering further $\alpha_2 = -1, y_2 = t, \alpha_1 = \alpha, y_1 = y$ one can find from (1.15) that the function $U(x,t,y)$ satisfies the Kadomtzev-Petviashwili (KP) equation

$$\frac{\partial}{\partial x}(\frac{\partial U}{\partial t} + 6U\frac{\partial U}{\partial x} + U_{xxx}) + 3\alpha^2 U_{yy} = 0 \tag{1.18}$$

In the case $\alpha^2 = -1$ it is the KP-1 equation, in the case $\alpha^2 = 1$ it is the KP-2 equation.

2 Two comments.

1. The developed theory is still correct if the operation $K * F$ is understood in more general sense:

$$K * F = (1-\mu)\int_x^\infty K(x,s)F(s,z)ds - \mu\int_{-\infty}^x K(x,s)F(s,z)ds \tag{2.1}$$

In particulary, instead of (1.1) we can start from the equation (2.1) with $\mu = 1$

$$K(x,z) + F(x,z) - \int_{-\infty}^x K(x,s)F(s,z)ds = 0 \tag{2.2}$$

or from the equation

$$K(x,z)+F(x,z)+\frac{1}{2}\int_x^\infty K(x,s)F(s,z)ds-\frac{1}{2}\int_{-\infty}^x K(x,s)F(s,z)ds = 0 \tag{2.3}$$

Varying μ at the same F we will get *different solutions* of the same nonlinear equation (1.15). In the matrix case μ could be replaced by a matrix, commuting with operator $L_0^{1,2}$ ($[L_0^{1,2}, \mu] = 0$).

2. The system (1.14) could be considered as an integrable nonlinear system instead of (1.15), if the unknown functions are the coefficients of operators $L^{1,2}$. The coefficients of "bare operators" $L_0^{1,2}$ satisfies the same system of equations (1.11). It means that we can do dressing procedure starting from any given solution of the nonlinear system (1.14). If the coefficients of operators $L_0^{1,2}$ are constant we speak about dressing over a trivial background. In an opposite case we have dressing over a nontrivial background. It was used first by E.A.Kuznetzov and A.V.Mikhailov [3] to describe a propagation of a soliton on a knoidal wave in the KdV equation. Nontrivial dressing for 2+1 dimension equations will be studied in the article of A.Fokas and the author [4].

3 The generalized N-wave equations.

Suppose now that we have *three or more* commuting dressible operator $D_i([D_i, D_j] = 0, i = 1, ..., n)$, $n \geq 3$. Suppose also that a function F satisfies the set of equations

$$D_i F = 0 \qquad i = 1, ...n. \tag{3.1}$$

After the dressing procedure we get the system of $\frac{1}{2}n(n-1)$ equations

$$\alpha_i \frac{\partial L^j}{\partial y_i} - \alpha_j \frac{\partial L^i}{\partial y_j} + [L^i, L^j] = 0 \tag{3.2}$$

Only $(n-1)$ of these equations are independent, all others are satisfied due to the Jacoby identity. The system (3.2) is overdeterminated but compatible. It could be used for constructing of new integrable systems. It is possible , for instance, to exclude all x-derivatives from the system (3.2). Let

$$\alpha_i = 1 (i = 1, 2, 3) \qquad L_0^i = I^i \frac{\partial}{\partial x} \qquad [I^i, I^j] = 0$$

Here I^i is a diagonal commuting matrix. It is easy now to get for dressed operators

$$L^i = I^i \frac{\partial}{\partial x} + [I^i, Q]$$

$$Q = K(x, y_1, y_2, y_3)$$

And the system (3.2) became the following

$$\frac{\partial}{\partial y_j}[I^i, Q] - \frac{\partial}{\partial y_i}[I^j, Q] + I^j \frac{\partial Q}{\partial x} I^i - I^i \frac{\partial Q}{\partial x} I^j + [[I^j, Q], [I^i, Q]] = 0 \tag{3.3}$$

Multiplying the equation (3.3) by I_k from the left side and doing the cyclic permutation we drop out all the terms containing $\frac{\partial Q}{\partial x}$, and get as a result

$$\varepsilon_{ijk}(I_i \frac{\partial Q}{\partial y_j} I_k - I_i Q I_j Q I_k) = 0 \tag{3.4}$$

Here ε_{ijk} is a totally antisymmetric unit tensor. The equation (3.4) is a natural generalization of so-called "N-wave" system. This equation is invariant under the transformation

$$I_i \rightarrow I_i + \alpha I_j + \beta I_k$$

If the matrix $I_k = 1$, the system (3.4) coincides to (3.3) after the change $\frac{\partial}{\partial y_k} \rightarrow \frac{\partial}{\partial x}$. For $N = 3$ we can put one of the matrixes I_i $(i = 1, 2, 3)$ to be scalar and get the equation (3.3). For $N \geq 4$ the system (3.4) is really more general then (3.3). About the system (3.3) see also [4].

4 Shock-wave on solution.

Let us now concentrate our attention on the application of the "old" dressing method to the equation KP-2

$$\frac{\partial}{\partial x}(U_t + U_{xxx} + 6UU_x) + 3U_{yy} = 0 \tag{4.1}$$

We will use the integral equation (1.1). The function F satisfies the system

$$\begin{aligned} \frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial z^2} &= 0 \\ \frac{\partial F}{\partial t} + 4(\frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial z^3}) &= 0 \end{aligned} \tag{4.2}$$

One can find the solution (4.2) in the form

$$F = \phi(x,y,t)\theta(z,y,t) \tag{4.3}$$

$$\frac{\partial \phi}{\partial y} + \frac{\partial^2 \phi}{\partial x^2} = 0 \qquad \frac{\partial \theta}{\partial y} - \frac{\partial^2 \theta}{\partial z^2} = 0 \tag{4.4}$$

$$\frac{\partial \phi}{\partial t} + 4\frac{\partial^3 \phi}{\partial x^3} = 0 \qquad \frac{\partial \theta}{\partial y} + 4\frac{\partial^3 \theta}{\partial z^3} = 0 \tag{4.5}$$

To provide a convergence of the integral in (1.1) let us demand F vanishing at $x \to \infty$, $z \to \infty$. We may find ϕ, θ in the form

$$\phi = \int_0^\infty f(\lambda) e^{-\lambda x - \lambda^2 y + 4\lambda^3 t} d\lambda \tag{4.6}$$

$$\theta = \int_0^\infty h(\lambda) e^{-\lambda y + \lambda^2 y + 4\lambda^3 t} d\lambda \tag{4.7}$$

The equation (1.1) could be solved easily

$$K(x,z) = \psi(x,y,t)\theta(z,y,t) \tag{4.8}$$

Here

$$\begin{aligned} \psi &= -\frac{\phi}{1 + \int_x^\infty \phi\theta dx} \\ U &= 2\frac{d}{dx}K(x,x) = -2\frac{d}{dx}(\frac{\phi\theta}{1 + \int_x^\infty \phi\theta dx}) \end{aligned} \tag{4.9}$$

This solution could be rewritten in the following form

$$U = 2\frac{d^2}{dx^2}\log R \tag{4.10}$$

$$R = 1 + \int_x^\infty \phi\theta dx \tag{4.11}$$

We found an exact solution of the KP-2 equation. Let the functions $f(\lambda), h(\lambda)$ now be *real and positively definite* and vanishing if $\lambda \to \infty$ faster than $e^{-A\lambda^2}$ at any A. Under this assumptions the solution (4.9) has no singularities on (x,y) plane. In the particular case $f(\lambda) = h(\lambda)$ this solution is symmetric with respect to changing sign of y : $U(x,y) = U(x,-y)$.

Choosing

$$f(\lambda) = h(\lambda) = \sqrt{2\nu}\delta(\lambda - \nu) \tag{4.12}$$

we have

$$U(x,t) = \frac{2\nu^2}{ch^2\nu(x - 4\nu^2 t)} \tag{4.13}$$

It is a solution of amplitude ν^2 moving in x-direction. It is very interesting to study a more general solution in some sense close to the soliton. Let us assume

$$f(\lambda) = h(\lambda) = \sqrt{2\nu}g(\lambda - \nu) \tag{4.14}$$

The function $g(\xi) > 0$ is concentrated on the support $-\Delta \leq \xi \leq \Delta$. At condition $\frac{\Delta}{\nu} \to 0$ we find

$$R = 1 + e^{-2\nu x'}\Phi(x' - 2\nu y - 4\nu^2 t)\Phi(x' + 2\nu y - 4\nu^2 t)$$

Here $x' = x - 4\nu^2 t$ is a coordinate in the moving frame, while

$$\Phi(z) = \int_{-\Delta}^{\Delta} g(\xi)e^{-\xi z}\partial\xi$$

The solution (4.9) describes the process at "straghtening" of initially curved solution, as it is drawn on the picture 1 (Section 5). This process is corresponded by propagation of two "shock-waves of compression", producing in the negative half-plane $x' < 0$ some extended weak radiation. To get a more general solution, let us put

$$F = \sum_{n=1}^{N} \phi_n(x,y,t)\theta_n(z,y,t) \tag{4.15}$$

$$\phi_n(x,y,t) = \int_0^\infty f_n(\lambda)e^{-\lambda x - \lambda^2 y + 4\lambda^3 t}d\lambda \tag{4.16}$$

$$\theta_n(x,y,t) = \int_0^\infty h_n(\lambda)e^{-\lambda x + \lambda^2 y + 4\lambda^3 t}d\lambda \tag{4.17}$$

The solution is given by the formula (4.10)

$$R = det \mid\mid \delta_{ij} + \int_x^\infty \phi_i\theta_i dx \mid\mid \tag{4.18}$$

This solution describes a nonlinear superposition on N curved and straghtening solutions.

One can use, instead of (1.1), the equation (2.2). In this case a simple solution has the form

$$U = 2\frac{d^2}{dx^2}\ln R$$
$$R = 1+\int_{-\infty}^{x}\tilde{\phi}\tilde{\theta}dx$$

Here

$$\tilde{\phi} = \int_0^{\infty} f(\lambda)e^{\lambda x-\lambda^2 y-4\lambda^3 t}d\lambda$$
$$\tilde{\theta} = \int_0^{\infty} h(\lambda)e^{\lambda x+\lambda^2 y-4\lambda^3 t}d\lambda$$

One can see that this equation could be obtained from the previous one by the single transformation $x \to -x, t \to -t$. Performing this transformation in (4.14), we have a more general solution

$$U = 2\frac{d^2}{dx^2}\log\det \| \delta_{i,j} + \int_{-\infty}^{x} \tilde{\phi}_i\tilde{\theta}_j\partial x \| \quad (4.19)$$

5 Towards the "new" dressing method.

The obtained solutions are interesting enough from the view point of physics. But they are very far from some kind of general solutions of KP-2. The general solution could not be found by means of the "old" dressing method. Any attempt to choose the function F, different from described above, gives the singular solution on (x,-y) - plane. It seems attractive to choose functions ϕ and θ in the form

$$\phi = \int_{-\infty}^{\infty} f(k)e^{ikx+k^2y-4ik^3t}dk \quad (5.1)$$
$$\theta = \int_{-\infty}^{\infty} h(k)e^{ikx-k^2y-4ik^3t}dk \quad (5.2)$$

But one can see that when $y \to \infty$ the function ϕ grows exponentially fast, while θ decreases slowly. As a result the determinant R becomes zero on some line $x = x_0(y)$, where the solution has severe singularities $U \sim c/(x-x_0(y))^2$. Using the previousely described version of the dressing method we could not find even such fundamental solutions of KP-2 that are small in some proper norm and close to solutions of the linearized equation.

It is clear that the dressing method should be seriously generalized. To do so we will start from the equation (2,3). (Note that the framework of the "old"

dressing method is almost useless, because it produces only singular solutio
To generalize the dressing method, we should release the restrictions on
asymptotic behavior of $F(x,z)$ at $x \to \pm\infty$, $z \to \pm\infty$. Let us study the opera

$$\partial^{-1} f(x) = \frac{1}{2}\int_{-\infty}^{x} f(x)dx - \frac{1}{2}\int_{x}^{\infty} f(x)\text{x} \qquad (5$$

This operator is well-defined not only if $f(x) \to 0$ at $x \to \pm\infty$ fast enough. L
$f(x)$ be presented by the Fourier transformation

$$f(x) = \int_{-\infty}^{\infty} f(k)e^{ikx}dk \qquad (5$$

In this case

$$\partial^{-1} f(x) = \fint_{-\infty}^{\infty} \frac{f(k)}{ik} e^{ikx}dk \qquad (5$$

Here $\fint_{-\infty}^{\infty}$ is a principal value of the integral. Assume now that $f(x)$ is rep
sented by the integral

$$f(x) = \int_{\Omega} f(k,\bar{k})e^{ikx}dk \qquad (5$$

Here Ω is some domain in k–plane. In the general position $f(x)$ grows expone
tially at $x \to \pm\infty$.

The operator (5.3)could be extended to functions (5.5) by the following w

$$\partial^{-1} = \fint_{\Omega} \frac{f(k,\bar{k})}{ik} e^{ikx}dkd\bar{k} \qquad (5.$$

The symbbol $\fint$ means that we understand $1/k$ as a limit

$$\frac{1}{k} = \lim_{\epsilon\to 0} \frac{\bar{k}}{|k|^2+\epsilon^2} \qquad (5.$$

In the particular case when Ω is a real axis, (5.6) coincides with (5.4). If th
function $f(k,\bar{k})$ is decreasing fast enough at $|\, k \,| \to \infty$, the whole k-plane cou
be chosen instead of Ω. The only thing that should be done to extend treme
dously the possibilities of the dressing method is to understand the integrals i
(2.3) as it was described above. Returning to the equation (2.3),let us represe
F and K in the following form

$$F = -\frac{1}{\pi}\int T(\nu,\bar{\nu},\lambda,\bar{\lambda})e^{i(\nu x-\lambda(x-z))}d\nu d\bar{\nu}d\lambda d\bar{\lambda} \qquad (5.9$$

$$K = -\frac{1}{\pi}\int Q(x,\lambda,\bar{\lambda})e^{i\lambda(x-z)}d\lambda d\bar{\lambda} \qquad (5.10$$

and denote that

$$G = Te^{i(\nu-\lambda)x} \qquad (5.11$$

Let us substitute(5.8) and (5.9) into (2.3) and do integration according to th
rule(5.6). The equation (2.3) is satisfied if T and Q obey the following equalit

$$Q(x,\lambda,\bar\lambda) - \int G(\nu,\bar\nu,\lambda,\bar\lambda)d\nu d\bar\nu + \frac{1}{\pi i}\int \frac{Q(x,\mu,\bar\mu)G(\nu,\bar\nu,\lambda,\bar\lambda)}{\nu-\mu}d\nu d\bar\nu = 0 \quad (5.12)$$

Introducing the function

$$\chi(\lambda,\bar\lambda) = 1 + \frac{1}{\pi}\int \frac{Q(x,\mu,\bar\mu)}{\lambda-\mu}d\mu d\bar\mu \quad (5.13)$$

and using the well-known formula

$$\frac{1}{\pi}\frac{\partial}{\partial\bar\lambda}\frac{1}{\lambda-\nu} = \delta(\lambda-\nu) \quad (5.14)$$

we get

$$Q(x,\lambda,\bar\lambda) = i\frac{\partial\chi}{\partial\bar\lambda}$$

Hence the equation (5.11) could be rewritten in a form

$$i\frac{\partial\chi}{\partial\bar\lambda} = \int \chi(x,\nu,\bar\nu)G(\nu,\bar\nu,\lambda,\bar\lambda)d\nu d\bar\nu \quad (5.15)$$

It means that the generalized equation (2.3) is equivalent to the equation which provides the solving of "nonlocal $\bar\partial$ – problem"(5.14). It is assumed that the $\bar\partial$ – problem is normalized by the condition

$$\chi \to 1 \qquad at \qquad |\lambda| \to \infty \quad (5.16)$$

The other possibility to solve this problem is using the equation on a function χ

$$\chi = 1 - \frac{i}{\pi}\int \frac{\chi(x,\nu,\bar\nu)G(\nu,\bar\nu,\mu,\bar\mu)}{\lambda-\mu}d\nu d\bar\nu d\mu d\bar\mu \quad (5.17)$$

In these cases we can assume that all functions are $N \times N$ matrix functions. If the function F satisfies the equation (3.1), and functions $l_k^i (i = 0, 1, ...)$ don't depend on x,the function G obeys the equation

$$\frac{\partial G}{\partial y_i} + L_i(i\nu)G - GL_i^+(i\lambda) = 0 \quad (5.18)$$

Here $L_i(i\nu)$ is the symbol of the operator L_i. As a function of parameter x, the function G obeys also the equation

$$\frac{\partial G}{\partial x} = i(\lambda-\nu)G \quad (5.19)$$

In particular, for the KP-2, we have

$$G = G_0(\nu,\bar\nu,\lambda,\bar\lambda)e^{i(\lambda-\nu)x+(\lambda^2-\nu^2)y-4i(\lambda^3-\nu^3)t} \quad (5.20)$$

Suppose $G_0(\nu,\bar\nu,\lambda,\bar\lambda) = G_0(\lambda,\bar\lambda)\delta(\lambda+\nu)\delta(\bar\lambda+\bar\nu)$. In this case the dependence on the variable y drops out, and we move from the KP-equation to the KdV-equation

$$U_t + 6UU_x + U_{xxx} = 0$$

Hence, we found that in order to solve the KdV equation we should solve the following $\bar{\partial}$- problem

$$i\frac{\partial \chi}{\partial \bar{\lambda}} = G_0(\lambda, \bar{\lambda})e^{2i\lambda x - 8ix^3 t}\chi(-\lambda, -\bar{\lambda}) \tag{5.21}$$

This fact was established in the article [2] and reopened after by many authors.

The function $K_0 = K(x, \lambda)$ is connected with the function Q by the formula

$$K_0 = \frac{1}{\pi i}\int Q(x, \mu, \bar{\mu})d\mu d\bar{\mu} \tag{5.22}$$

There is one obvious difference between the old and the new versions of the dressing method. The function F has two independent variables — as arbitrary as the general solution of an evolution equation . On the contrary , the function G has four independent variables $\lambda, \bar{\lambda}, \nu, \bar{\nu}$. It leads to a conjecture that this version of the dressing method could be used for solving an equation in more than $2+1$ dimensions. This is an illusion . The different choice of T could lead to a different Q but to the same K_0 and to the same solution of the equation. Only the function F is important, it means that the function T is defined up to adding an arbitrary $\tilde{T}$

$$T \to T + \tilde{T} \tag{5.23}$$

where

$$\int \tilde{T}(\nu, \bar{\nu}, \lambda, \bar{\lambda})e^{i(\nu x - \lambda z)}d\nu d\bar{\nu} d\lambda d\bar{\lambda} = 0 \tag{5.24}$$

Note, that if Ω in (5.5) is bounded domain with boundary Γ, an integral on the domain could be replaced by the integral over Γ

$$\int_\Omega f(k, \bar{k})e^{ikx}dx = \frac{1}{2\pi i}\int_\Gamma \tilde{f}(\xi, \bar{\xi})e^{i\xi x}d\xi \tag{5.25}$$

where

$$\tilde{f}(\xi, \bar{\xi}) = \int_\Omega \frac{f(k, \bar{k}}{\xi - k}dkd\bar{k} \tag{5.26}$$

A similar formula is also correct for F.

6 The new dressing method

In the advanced variant of the dressing method we starts directly from the $\bar{\partial}$-problem (5.14) with the condition of normalization (5.15).We will assume this problem to be solvable by the unique way.It means that the function χ obeying equation (5.14) and vanishing at large λ

$$\chi(\lambda) \to 0 \qquad |\lambda| \to \infty \tag{6.1}$$

is identically equal to zero : $\chi(\lambda) = 0$.

Let us introduce three commuting first order operators $D_i (i = 1, 2, 3)$

$$D_i \chi = \frac{\partial \chi}{\partial y_i} + i \chi I_i(\lambda) \tag{6.2}$$

using three commuting $[I_i(\lambda), I_j(\lambda)] = 0$ matrix functions of the spectral parameter λ. In the first step we will assume them to be polynomial. The function G could be consider as the kernel of the integral operator $\hat{G}$, required to commute with D_i. This requirement imposes on G the system of equations

$$\frac{\partial G}{\partial y_i} = i(I_i(\nu) G - G I_i(\lambda)) \tag{6.3}$$

This system could be solved

$$G = e^{i\Phi(\nu)} G_0(\nu, \bar{\nu}, \lambda, \bar{\lambda}) e^{-i\Phi(\lambda)} \tag{6.4}$$

Here $\Phi(\lambda) = \sum_k I_k(\lambda) y_k$.

Let us rewrite (5.14) in a symbolic form

$$i \frac{\partial \chi}{\partial \bar{\lambda}} = \chi * G \tag{6.5}$$

Polyvomial on λ operators D_i, commute with the operators $\partial / \partial \bar{\lambda}$. Due to (6.3) one has

$$i \frac{\partial}{\partial \bar{\lambda}} D_i \chi = D_i \chi * G \tag{6.6}$$

Let $R = R(D_1, D_2, D_3)$ be any differential operator polynomial on D_i. Its coefficients are a matrix function of y_i acting on χ from the left side. Obviously $R \partial / \partial \bar{\lambda} = \partial / \partial \bar{\lambda} R$, we have

$$i \frac{\partial}{\partial \bar{\lambda}} R \chi = R \chi * G \tag{6.7}$$

The function $R\chi$ has at $\lambda \to \infty$ a polynomial-type behavior and in general it is not equal to zero. *But for any given function* χ it is possible to find a set of operators $\tilde{R}$ such that $\tilde{R}\chi \to 0$ at $|\lambda| \to \infty$. It means that

$$\tilde{R} \chi = 0 \tag{6.8}$$

It is obvious that the operator $\tilde{R}$ constitute a left ideal in the noncomutative ring of operators R. It was shown in [2] that it is possible to find two linearly independent operators $\tilde{R}_{1,2}$ serving as a basis in $\tilde{R}$. The function χ nearby the infinite point could be represented by the expansion

$$\chi = 1 + \frac{\chi_0}{\lambda} + \frac{\chi_1}{\lambda^2} + \dots \tag{6.9}$$

The coefficients of $\tilde{R}_{12}$ are polynomial on χ_i and on their derivatives. Let us consider the very simple example

$$I_k(\lambda) = I_k \lambda \qquad [I_i, I_j] = 0 \tag{6.10}$$

Now

$$D_i \chi = \frac{\partial \chi}{\partial y_i} + \chi I_i \lambda \tag{6.11}$$

Let us introduce the operators

$$L_{ij} = I_i D_j - I_j D_i - Uij \tag{6.12}$$

$$U_{ij} = i(I_i \chi_0 I_j - I_j \chi_0 I_i) \tag{6.13}$$

One can see that $L_{ij}\chi \sim 0(\frac{1}{\lambda})$.Hence

$$L_{ij}\chi \equiv 0 \tag{6.14}$$

With order $\frac{1}{\lambda}$ we have from (6.14)

$$I_j \chi_1 I_i - I_i \chi_1 I_j = I_i \frac{\partial \chi_0}{\partial y_j} - I_j \frac{\partial \chi_0}{\partial y_i} - U_{ij}\chi_o \tag{6.15}$$

Multiplying from the right side by I_k and performing a cyclic permutation we drop out all the members containing χ_1. As a result, we get

$$\varepsilon_{ijk}(I_i \frac{\partial \chi_0}{\partial y_j} I_k - i I_i \chi_0 I_j \chi_0 I_k) = 0 \tag{6.16}$$

One can see that we get the equation (3.4) where $Q = i\chi_0$.

The question arises — how many of the new integrable equations one can find using the new dressing method instead of the old one. Note that if one of the functions I_i is scalar and linear ($I_3 = \lambda$)we get the complete set of equations that is given by the old method. In the general case we could get some new equations . But the comparision of equations (3.4) and (6.16) show that using the procedure of excluding x–derivatives from an initially overdeterminated system, as was described in section 3, one can get by the old method the same equations that may be obtained by the new one.

The great advantage of the new method is the possibility to put functions I_1, I_2, I_3 be rational instead of polynomial. In this case

$$\frac{\partial}{\partial \bar{\lambda}} D_i \neq D_i \frac{\partial}{\partial \bar{\lambda}}$$

If for instance

$$I_i(\lambda) = \frac{I_i}{\lambda - \lambda_i}$$

then

$$\frac{\partial}{\partial \bar{\lambda}} D_i - D_i \frac{\partial}{\partial \bar{\lambda}} = \pi I_i \delta(\lambda - \lambda_i) \tag{6.17}$$

For a general operator $R = R(D_1, D_2, D_3)$ the commutator

$$\frac{\partial}{\partial \bar{\lambda}} R - R \frac{\partial}{\partial \bar{\lambda}} = \Delta(\lambda) \tag{6.18}$$

is not equal to sero. In general $\Delta(\lambda)$ is a sum of δ–like singularities in all poles of R.

In order to find the operator $\tilde{R}$ (6.8) obeying the equation (6.18) one should require $\tilde{R}\chi \to 0$ at $\lambda \to \infty$ and

$$\tilde{R} \frac{\partial}{\partial \bar{\lambda}} = \frac{\partial}{\partial \bar{\lambda}} \tilde{R}$$

It was shown in [2] that both these conditions could be satisfied for at least two linearly independent operators $\tilde{R}_1, \tilde{R}_2$. These operators are direct analogs of "dressed" operators L_1, L_2 in the old version of the method. See also [11,12].

7 Small solutions and dispersion laws

In the old dressing method the variable x is the special one. In the new method all the variables y_i have the same abilities. In the old method we must consider the global behavior of the solution on the x-axis, in particular at $x \to \pm\infty$. That is an inheritage of the Inverge Scattering Transform. The new version of the method cuts any connections with the Inverge Scattering Approach. A solution obtained due to the $\bar{\partial}$-problem is essensially *local* in the space (y_1, y_2, y_3).In general it could grow arbitrarily fast at $|y| \to \infty$. Moreover, it could have singularities (as described in section 5) on some set S in the space (y_1, y_2, y_3). In the general position this set is a surface, it is obviousely that S coincides with the set where the $\bar{\partial}$-problem (5.14) is unresolvable. The question arises —how to choose the "dressing function" $G_0(\nu, \bar{\nu}, \lambda, \bar{\lambda})$ to produce a "good enough" solution of the nonlinear system. In order to formulate the problem more accurately, let us suppose that one of the variables is "time"($y_3 = t$, for instance), while two others (y_1, y_2) are "spatial" variables. In other words , we consider our nonlinear system as an evolutional one. (Note, that a corresponding Cauchy problem may be ill-posed.)

Now let us try to find G_o in a such way that the initial data (at $y_3 = 0$) are regular in the plane (y_1, y_2) and vanish at $|y| =\!\!\to \infty$. This problem could easily be solved in the limit $G_0 \to 0$. In this case the solution of $\bar{\partial}$– problem is given by the first iteration of the integral equation (5.17). We have

$$\chi = 1 + \frac{1}{\pi i} \int \frac{G(\nu, \bar{\nu}, \xi, \bar{\xi})}{\lambda - \xi} d\nu d\bar{\nu} d\xi d\bar{\xi} \tag{7.1}$$

Comparing (7.1) and (6.9) one can find

$$\chi_n = \frac{1}{\pi i} \int \xi^n G(\nu, \bar{\nu}, \xi, \bar{\xi}) d\nu d\bar{\nu} d\xi d\bar{\xi} \tag{7.2}$$

Assuming that the matrixes $I_i(\lambda)$ are diagonal

$$I_i^{\alpha\beta}(\lambda) = \delta_{\alpha\beta} I_i^{\alpha}(\lambda)$$

we have for $\chi_n^{\alpha\beta}$

$$\chi_n^{\alpha\beta} = \frac{1}{\pi i} \int \xi^n G_0^{\alpha\beta}(\nu, \bar{\nu}, \xi, \bar{\xi}) e^{i \sum_{k=1}^{3} (I_k^{\alpha}(\nu) - I_k^{\beta}(\xi)) y_k} d\nu d\bar{\nu} d\xi d\bar{\xi} \tag{7.3}$$

To provide the regularity of $\chi_n^{\alpha\beta}$ at $y_3 = 0$ it is enough to require

$$Im(I_1^{\alpha}(\nu) - I_1^{\beta}(\xi)) \equiv 0 \tag{7.4}$$

$$Im(I_2^{\alpha}(\nu) - I_2^{\beta}(\xi)) \equiv 0 \tag{7.5}$$

These conditions define a two-dimensional manifold $\Lambda^{\alpha\beta}$ in four-dimensional space $\nu_R, \nu_I, \xi_R, \xi_I$ ($\xi = \xi_R + i\xi_I, \nu = \nu_R + i\nu_I$). The function $G_0^{\alpha\beta}(\nu, \bar{\nu}, \xi, \bar{\xi})$ should be concentrated on this manifold. If $I_{1,2}^{\alpha}(\nu)$ are real at real ν, the manifold $\Lambda^{\alpha\beta}$ includes the product of two real axis $R^1 \times R^1$. It could also include other components. Let us consider some simple examples .

1. KP-1.

$$\alpha = \beta = 0 \qquad I_1(\nu) = \nu \qquad I_2(\nu) = \nu^2$$

From (7.4), (7.5) one can get

$$\nu_R = \xi_R = 0$$

and

$$G_0(\nu, \bar{\nu}, \xi, \bar{\xi}) = G_0(\nu_R, \xi_R) \delta(\nu_I) \delta(\xi_I)$$

So the manifold Ω is the product of two real axes. The function χ is analytical in both half-planes $Im > 0, Im < 0$ and have a jump on the real axis. The equation (5.14) describes a nonlocal Riemann problem on the real axis.

2. KP-2.

$$\alpha = \beta = 0 \qquad I_1(\nu) = \nu \qquad I_2(\nu) = \alpha\nu^2 \qquad \alpha = \pm i$$

From (7.4), (7.5) one can get

$$\xi_I = \nu_I \qquad \xi_R = -\nu_R \qquad or \qquad \xi = -\bar{\nu} \tag{7.6}$$

So

$$G_0(\nu,\bar\nu,\xi,\bar\xi) = G_0(\nu,\bar\nu)\delta(\xi+\bar\nu) \tag{7.7}$$

Instead of (5.14) we now have

$$i\frac{\partial\chi}{\partial\bar\lambda} = G_0(\lambda,\bar\lambda)\chi(-\bar\lambda)e^{i(\lambda+\bar\lambda)x-(\lambda^2-\bar\lambda^2)y+4i(\lambda^3+\bar\lambda^3)t} \tag{7.8}$$

This result was first obtained in [5].

The formula (7.3) allows to solve one rather important problem. All nonlinear equations under consideration have one trivial solution

$$\chi_n \equiv 0 \qquad (n = 0, ...)$$

These equations could be linearized in the vicinity of this solution. After the linearization one can get N^2 independent linear equations with constant coefficients for different elements of the matrix $\chi_i^{\alpha\beta}$.These equations could be solved by the Fourier transformation method. Assuming

$$\chi_i^{\alpha\beta} \cong e^{i(py_1+qy_2+\omega y_3)} \tag{7.9}$$

one should find N^2 relations

$$R_{\alpha\beta}(\omega,p,q) = 0 \tag{7.10}$$

The formula (7.3) shows that this relation could be found directly before calculating the explicit form of a nonlinear system. Supposing that (7.4) and (7.5) are satisfied, one can find (7.9)in a parametric form

$$p = Re(I_1^\alpha(\nu) - I_1^\beta(\xi)) \tag{7.11}$$
$$q = Re(I_2^\alpha(\nu) - I_2^\beta(\xi)) \tag{7.12}$$
$$\omega = I_3^\alpha(\nu) - I_3^\beta(\xi) \tag{7.13}$$

Equations (7.10)—(7.12) show that $R_{\alpha\beta} = R_{\beta\alpha}$ and we have $N(N+1)/2$ independent relations. Expressing ν and ξ from (7.10),(7.11) and substituting them into (7.12) one can find the full set of "dispersion laws".

$$\omega = \omega_{\alpha\beta}(p,q) = \omega_{\beta\alpha}(p,q) \tag{7.14}$$

Finding the whole set of dispersion laws hidden in a given nonlinear system is the first step for this system to find any applications (see also [6]).

8 On extention of the dressing method

We shall start from the $\bar\partial$-problem

$$i\frac{\partial\chi}{\partial\bar\lambda} = \chi * G = \int \chi(\nu,\bar\nu)G(\nu,\bar\nu,\lambda\bar\lambda)d\nu d\bar\nu \tag{8.1}$$

and introduce a set of operators

$$D_i\chi = \frac{\partial\chi}{\partial y_i} + \chi * V_i \qquad i = 1, ..., n \tag{8.2}$$

Operators V_i in (8.2) could be some integral operators in general. In the general case their kernels $V_i(\nu, \bar{\nu}, \lambda, \bar{\lambda}, y_1...y_n)$ depend on the variables y_i. In the previous case we have

$$V_i = I_i(\lambda)\delta(\lambda - \nu)\delta(\bar{\lambda} - \bar{\nu}) \tag{8.3}$$

Now, suppose that G satisfies the system

$$\frac{\partial G}{\partial y_i} + G * V_i - V_i * G = 0 \tag{8.4}$$

This system is assumed to be compatible. A sufficient (perhabs, not necessary) condition is commutativity for the operators

$$[D_i, D_j] = 0 \tag{8.5}$$

We now have

$$iD_k\frac{\partial\chi}{\partial\bar{\lambda}} = D_k\chi * G \tag{8.6}$$

Again, let us consider that $R = R(D_1, ...D_n)$ is polynomial on D_i. Its coefficients are the $N \times N$ matrix function on $y_1, ...y_n$. Obviously

$$iR\frac{\partial\chi}{\partial\bar{\lambda}} = R\chi * G \tag{8.7}$$

suppose we have found an operator R satisfying the conditions

$$\tilde{R}\frac{\partial\chi}{\partial\bar{\lambda}} = \frac{\partial}{\partial\bar{\lambda}}\tilde{R}\chi \qquad R\chi \to 0 \qquad at|\lambda| \to \infty \tag{8.8}$$

Assuming that $\bar{\partial}$–problem (7.1) has a unique solution,we now have

$$\tilde{R}\chi \equiv 0 \tag{8.9}$$

The condition (8.8) could be satisfied for only for concrete χ. So coefficients of R depend on χ and (8.9) is a nonlinear equation for the function χ, having a known solution, depending on λ. To exclude this dependence one should have a set of equations

$$\tilde{R}_i\chi = 0 \qquad i = 1, ...k \qquad 2 \le k \le n \tag{8.10}$$

It is obvious that all the equations (8.10) are compatible.

It is hard to develop this theory for general integral operators. So we shall assume that V_i are *differential* operators on λ with *polynomial* on λ coefficients.

Even this very special case is not so simple. Let us consider an example (N=1)

$$D_1\chi = \frac{\partial \chi}{\partial x} + iV\chi \qquad V = \lambda^2 \frac{\partial}{\partial \lambda} \tag{8.11}$$

$$D_2\chi = \frac{\partial \chi}{\partial y} + V^2\chi \tag{8.12}$$

It is possible to construct two compatible equations

$$R_1\chi = \frac{\partial \chi}{\partial x} + i\lambda^2\frac{\partial \chi}{\partial \lambda} + i\chi_0\chi \equiv 0 \tag{8.13}$$

$$R_2\chi = (D_2 + D_1^2 + 2i\frac{\partial \chi_0}{\partial x})\chi \equiv 0 \tag{8.14}$$

The last equation may be rewritten in the form

$$\frac{\partial \chi}{\partial y} + \frac{\partial^2 \chi}{\partial x^2} + 2i\frac{\partial^2 \chi}{\partial \lambda \partial x} + 2i\frac{\partial \chi_0}{\partial x}\chi = 0 \tag{8.15}$$

Let us consider $\lambda \to \infty$ and use the expansion (6.9). In the order $\frac{1}{\lambda}$ we have

$$2i\chi_1 = \frac{\partial \chi_0}{\partial x} + i\chi_0^2$$

$$\frac{\partial \chi_0}{\partial y} + \frac{\partial^2 \chi_0}{\partial x^2} - 2i\frac{\partial \chi_1}{\partial x} + 2i\chi_0\frac{\partial \chi_0}{\partial x} = 0$$

or

$$\frac{\partial \chi_0}{\partial y} = \frac{\partial^2 \chi_0}{\partial x^2} + 2i\chi_0\frac{\partial \chi_0}{\partial x} \tag{8.16}$$

It means that function $W = i\chi_0$ satisfies the Burgers equation. Adding to (8.11),(8.12) the third operator

$$D_3\chi = \frac{\partial \chi}{\partial t} + 4iV^3\chi \tag{8.17}$$

one can find that χ_0 satisfies two other Burgers-like equations. It is interesting that at the same time the function $U = 2i\frac{\partial \chi_0}{\partial x}$ satisfies the KP-2 equation.

In the previous example, D_i belongs to the simplest class of commuting differential operators with polynomial coefficients

$$D_i^0\chi = \frac{\partial \chi}{\partial y_i} + P_i(V_0)\chi \tag{8.18}$$

Here P_i are polynomials on some elementary operator V_0 with constant coefficients. In the matrix case, all coefficients of P_i must commute with V_0. To get more general polynomial differential operators one could apply to D_i^0 a gauge transformation

$$D_i = e^{-i\Phi}D_i^0e^{i\Phi} \tag{8.19}$$

Here Φ is polynomial on λ with y_i independent coefficients. This case includes the previously described (in section 6) variant of the dressing method. Assuming

$$D_i^0 = \frac{\partial}{\partial y_i} \qquad \Phi = \sum I_i(\lambda) y_i \tag{8.20}$$

one can get

$$D_i \chi = \frac{\partial \chi}{\partial y_i} + i\chi I_i(\lambda) \qquad i = 1, 2, 3 \tag{8.21}$$

To obtain a more advanced example, put in the $N \times N$ matrix case

$$D_i^0 \chi = \frac{\partial \chi}{\partial y_i} + \frac{\partial \chi}{\partial \lambda}(1 + \alpha\lambda) A_i \tag{8.22}$$

$$i = 1, 2, 3 \qquad [A_i A_j] = 0 \qquad [A_i, \alpha] = 0$$

After the transformation (8.19), using

$$\Phi = \lambda \sum I_i y_i \qquad [A_i, I_j] = 0 \qquad [I_i, I_j] = 0$$

we have

$$\tilde{D}_i \chi = D_i \chi - i\Phi A_i \chi = \frac{\partial \chi}{\partial y_i} + \frac{\partial \chi}{\partial \lambda}(1 + \alpha\lambda) A_i + i\lambda\chi \tilde{I}_i + i[\chi_1 \Phi A_i] \tag{8.23}$$

$$\tilde{I}_i = I_i + \alpha \Phi A_i$$

It is easy to show that

$$L_{ij} \chi = 0 \tag{8.24}$$

with

$$\begin{aligned} L_{ij} &= \tilde{I}_i \tilde{D}_j - \tilde{I}_j \tilde{D}_i - U_{ij} \\ U_{ij} &= i(\tilde{I}_i \chi_0 \tilde{I}_j - \tilde{I}_j \chi_0 \tilde{I}_i) \end{aligned} \tag{8.25}$$

Acting like we did in section 6 one can get the equation for χ_0

$$\varepsilon_{ijk}(\tilde{I}_i \frac{\partial \chi_0}{\partial x_j} \tilde{I}_k - i\tilde{I}_i \chi_0 \tilde{I}_j \chi_0 \tilde{I}_k + i\tilde{I}_i [\chi_1 \Phi A_j] \tilde{I}_k - \alpha \tilde{I}_i \chi_0 A_j \tilde{I}_k) = 0 \tag{8.26}$$

This equation is a generalization of the N-wave system (3.4),(6.16). Matrixes $\tilde{I}_i$ are now linear functions of coordinates. The system (8.26) may have some physical applications. Using the operators (8.18), (8.19) we can get a very broad class of "integrable" systems. To show that , let us consider the operators (8.20), (8.21) producing some nonlinear integrable systems, as it was described in section 6. Let us add to the operator D_i^0 another one operator

$$D_1^0 \chi = \frac{\partial \chi}{\partial \tau} + V_0 \chi \tag{8.27}$$

Here V_0 is any differential operator

$$V_0\chi = \frac{d^2\chi_0}{\partial\lambda^n} l_0 + \dots$$

commuting with the matrixes I_i.

Let us also perform transformation

$$\Phi \to \tilde{\Phi} = \Phi + P(\lambda)\tau \qquad [P(\lambda), I_i] = 0$$

Here $P(\lambda)$ is an arbitrary polynomial. Now the operator

$$D_4 = e^{-i\tilde{\Phi}} D_4^0 e^{i\tilde{\Phi}} \tag{8.28}$$

commutes with operators (8.21). Using any triplet of operators (D_1, D_2, D_4),(D_1, D_3, D_4),(D_2, D_3, D_4) as a basic system one can get three new nonlinear systems, compatible with the initial equations. All of them are *symmetries* of the initial systems. If $V_0 \neq 0$, these symmetries have coordinate-depending coefficients. It is plausible that on this way one may find all the symmetries of $2+1$ dimension integrable systems.

9 Some perspectives

A further developing of the dressing method depends upon a progress in the classification of the operators D_i. Besides the class of these operators, described in the section 8, one can separate two other natural classes.

1. The class of operators D_i not including λ-derivatives

$$D_i\chi = \frac{\partial\chi}{\partial y_i} + \chi U_i(\lambda, y_i) \qquad i = 1, 2, 3 \tag{9.1}$$

Here $U_i = U_i(\lambda, y)$ are rational functions on λ with fixed poles. In fact, poles of U_i may depend on y_i. The commutativity condition $[D_i D_j] = 0$ gives

$$\frac{\partial U_i}{\partial y_j} - \frac{\partial U_j}{\partial y_i} + [U_j, U_i] = 0 \tag{9.2}$$

The equations (9.2) could be solved by $1+1$ variant of the dressing method, using a local $\bar{\partial}$-problem or a local Riemann problem. This variant of the dressing method was described in article [7].

2. The class of operators D_i including only first λ-derivatives with a scalar coefficients.

$$D_i\chi = \frac{\partial\chi}{\partial y_i} + \frac{\partial\chi}{\partial\lambda} W_i + \chi U_i \tag{9.3}$$

Here $U_i = U_i(\lambda, y)$ are matrixes, while $W_i = W_i(\lambda, y)$ are scalar rational on λ functions. They have the same (for a given i) y-dependent poles. The commutativity conditions impose the well definite system of equations on this poles and on the residues of W_i and U_i ([8], see also [9]). This system also could be solved by some extention of the dressed method, based on the local $\bar{\partial}$-problem. This variant of the method was called in article [8] "The Inverse Scattering Method with a Moving Spectral Parameter".

To go further we should consider that $W_i(\lambda, y)$ are matrix functions, or introduce λ-derivatives of a higher order. The only known class of such D_i-operators was described in section 8. To find more general classes we should use the old dressing method, maybe in its modernized form, introduced in this article. Let us suppose that the variable x is in fact a spectral parameter ($x = \lambda$). To find the D_i-operators we should find a solutions of the system (3.2) *that are rational functions of* x. In general, their poles depend on variables y_i. To classify the solutions of this type is a very intriguing mathematical problem. For the KP-equation this problem was solved by Krichever (see, for instance [10]). It was shown that in the y-dependent case, the system governing moving of poles coincides with the well-known Calogero– Moser system of particle of the axis with pair interaction $U_{ij} \sim \frac{c}{(x_i - x_j)^2}$.

Let us do one concluding remark. A mathematician, using the dressing method to find a new integrable system, could be compared with a fishman, plunging his net into the sea. He does not know what a fish he will pull out. He hopes to catch a goldfish, of course. But too often his catch is something that could not be used for any known to him purpose. He invents more and more sophysticated nets and equipments and plunge all that deeper and deeper. As a result he pulls on the shore after a hard work more and more strange creatures. He should not despair, nevertheless. The strange creatures may be interesting enough if you are not too pragmatic. And who knows how deep in the sea do goldfishes live?

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