Math 422

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Homework 5. Due May 7 2019

1. Determine Fourier Transform

$$\hat{F}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-ikx}\,dx$$ of the function

$$f(x) = \begin{cases} \cos ax & |ax| < \frac{\pi}{2} \\ 0 & |ax| > \frac{\pi}{2} \end{cases}$$

2. By the use of Fourier Transform solve the Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0$$ inside the strip $0 < y < a$

with boundary conditions

$$u\big|_{y=0} = 0 \qquad u\big|_{y=a} = f(x)$$

3. Solve the wave equation

$$u_{tt} = c^2 (u_{xx} + u_{yy})$$

in the same strip $0 < y < a$

with zero boundary conditions on both sides of the strip.

The initial data are

$$u\big|_{t=0} = f(x) \sin \frac{\pi}{a} y \qquad u_t\big|_{t=0} = 0$$

4. Solve the following problems by the use of Laplace transforms

(a) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 2y = 0 \qquad y(0) = 1 \quad \dfrac{dy(0)}{dt} = -1$

(b) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 2y = \delta(t-1) \qquad y(0) = 1 \quad \dfrac{dy(0)}{dt} = -1.$

5. If $f(t)$ is the staircase function such that $f(t) = b$ when $0 < x < a$, $f(t) = 2b$ when $a < x < 2a$ and so forth, show that

$$\mathcal{L}\{f(t)\} = \frac{b}{s(1 - e^{-as})}$$

The wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} \right)$$

is defined inside the disk with boundary conditions

$$u\Big|_{r=R} = 0$$

R is radius of the disk

Determine first six eigenfrequences and eigenfunctions

(Hint — use my lecture notes on Bessel functions)