Math. 422-527

Istructor Vladimir Zakharov

Practice test for final exam

1. Solve equation

$$w^8 = 1$$

Solution: Equation has eight roots. They are

$$w_n = e^{\frac{2\pi i n}{8}} = e^{\frac{\pi i n}{4}} \qquad n = 0, \dots \cancel{7})$$

$$w_0 = 1, \quad w_1 = \frac{1}{\sqrt{2}}(1+i) \quad w_2 = i \quad w_3 = \frac{1}{\sqrt{2}}(-1+i) \quad w_4 = -1$$

$$w_5 = \frac{1}{\sqrt{2}}(-1-i) \quad w_6 = -i \quad w_7 = \frac{1}{\sqrt{2}}(1-i)$$

2. Find first three Legendre's polynomial

Solution. One should use the Rodriges formula

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2-1)^n$$

By the use of this formula one easy find

$$P_0 = 1 \qquad P_1 = x \qquad P_2 = \frac{1}{2}(3x^2-1)$$

(3) Present the function $f(x) = \sin x \quad 0 < x < \pi$ as a sum of cosine Fourier series

Solution

$$\sin x = a_0 + \sum a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^\pi \sin x\, dx = \frac{1}{2}$$

$$a_n = \frac{2}{\pi}\int_0^\pi \sin x \cos nx\, dx = \frac{1}{\pi}\int_0^\pi [\sin(1+n)x + \sin(1-n)x]\, dx$$

$$= -\frac{1}{\pi}\left[\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\right]_0^\pi$$

now mention that

$$\cos(1+n)\pi = \cos(1-n)\pi \Rightarrow (-1)^{1+n}$$

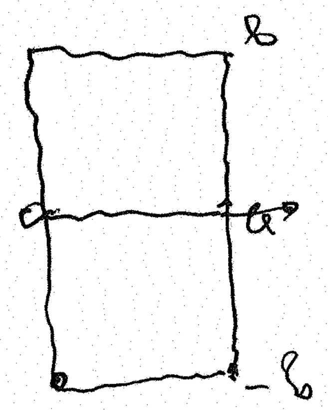
Then finally

$$a_n = \frac{2}{\pi}\,\frac{1}{1-n^2}\left\{1-(-1)^{1+n}\right\}$$

$a_n = 0$ for odd n.

4. Solve the Laplace equation for rectangular domain

$\Delta u = 0$



$0 < x < a$

$-b < y < b$

with boundary condition

$$u\big|_{x=0} = u\big|_{x=a} = 0$$

$$u\big|_{y=b} = u\big|_{y=-b} = \cancel{\cos}\ \sin\frac{\pi n}{a}x$$

The solution is following

$$u = \frac{\sin \frac{\pi n}{a} x \cosh \frac{\pi n}{a} y}{\cosh \frac{\pi n}{a} b}$$

5. Solve the heat equation

$$u_t = u_{xx}$$

on the whole axis $-\infty < x < \infty$

and initial date

$$u\big|_{t=0} = A\left[\delta(x-a) + \delta(x+a)\right]$$

Solution

$$u = \frac{A}{\sqrt{4\pi t}}\left[e^{-\frac{(x-a)^2}{4t}} + e^{-\frac{(x+a)^2}{4t}}\right]$$

6. Find the fourier transform of function

$$f(x) = \begin{cases} \sin x & |x| < \pi \\ 0 & |x| > 0 \end{cases}$$

Solution

$$\hat{F}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin x \, e^{-ikx}\,dx = \frac{1}{4\pi i}\int_{-\pi}^{\pi}\left[e^{-i(k-1)x} - e^{-i(k+1)x}\right]$$

$$= \frac{1}{4\pi}\left[\frac{e^{-i(k-1)x}}{k-1} - \frac{e^{-i(k+1)x}}{k+1}\right]_{-\pi}^{\pi} =$$

$$= \frac{i}{2\pi}\,\frac{\sin \pi k x}{k^2-1}$$

7. Find Laplace transform for the fuction

$$f(x) = x\cos x\, e^{-ax}$$

Solution

$$\mathcal{L}\,\cos x = \frac{s}{s^2+1}$$

$$\mathcal{L}\,x\cos x = \frac{d}{ds}\,\frac{s}{s^2+1} = \frac{1-2s^2}{(1+s^2)^2}$$

$$\mathcal{L}\,x\cos x\, e^{-at} = \frac{1-2(s+a)^2}{\left[1+(s+a)^2\right]^2}$$