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Sato theory

(1) Binomial coefficients

$$\binom{n}{i} = \frac{n(n-1)\dots(n-i+1)}{i(i-1)\dots 1} = \frac{n(n-1)\dots(n-i+1)}{i!}$$

$$\binom{n}{1} = n$$

$$\binom{-n}{1} = -n$$

$$\binom{n}{2} = \frac{n(n-1)}{1 \cdot 2}$$

$$\binom{-n}{2} = + \frac{n(n+1)}{2}$$

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{3!}, \quad \binom{-n}{3} = - \frac{n(n+1)(n+2)}{6}$$

$$\binom{-n}{4} = \frac{n(n+1)(n+2)(n+3)}{24}$$

$$\binom{-n}{5} = - \frac{n(n+1)(n+2)(n+3)(n+4)}{120}$$

Thus we get

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = -1 \quad \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix} = 1 \quad \begin{pmatrix} -1 & 1 \\ 3 & 3 \end{pmatrix} = -1 \quad \begin{pmatrix} -1 & 1 \\ 4 & 4 \end{pmatrix} = 1$$

$$\begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} = -2 \quad \begin{pmatrix} -2 & 2 \\ 2 & 2 \end{pmatrix} = 3 \quad \begin{pmatrix} -2 & 2 \\ 3 & 3 \end{pmatrix} = -4 \quad \begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix} = 5$$

$$\begin{pmatrix} -3 & 3 \\ 1 & 1 \end{pmatrix} = -3 \quad \begin{pmatrix} -3 & 3 \\ 2 & 2 \end{pmatrix} = 6 \quad \begin{pmatrix} -3 & 3 \\ 3 & 3 \end{pmatrix} = -9 \quad \begin{pmatrix} -3 & 3 \\ 4 & 4 \end{pmatrix} = 12$$

$$\begin{pmatrix} -4 & 4 \\ 1 & 1 \end{pmatrix} = -4 \quad \begin{pmatrix} -4 & 4 \\ 2 & 2 \end{pmatrix} = 10 \quad \begin{pmatrix} -4 & 4 \\ 3 & 3 \end{pmatrix} = -20 \quad \begin{pmatrix} -4 & 4 \\ 4 & 4 \end{pmatrix} = 30$$

$$\textcircled{2} \quad \partial^1 u = u \partial^{-1} - \partial u \partial^{-2} + \partial^2 u \partial^{-3} - \partial^3 u \partial^{-4} + \partial^4 u \partial^{-5}$$

$$\partial^2 u = u \partial^{-2} - 2 \partial u \partial^{-3} + 3 \partial^2 u \partial^{-4} - 4 \partial^3 u \partial^{-5}$$

$$\partial^3 u = u \partial^{-3} - 3 \partial u \partial^{-4} + 6 \partial^2 u \partial^{-5}$$

$$\partial^4 u = u \partial^{-4} - 4 \partial u \partial^{-5}$$

Let L is a forward operator

$$L = \partial + f_{-1} \partial^{-1} + f_{-2} \partial^{-2} + f_{-3} \partial^{-3} + \dots$$

$$L^2 = \partial^2 + u_0 + u_{-1} \partial^{-1} + u_{-2} \partial^{-2} + \dots$$

$$L^3 = \partial^3 + v_1 \partial + v_0 + v_{-1} \partial^{-1} + \dots$$

The problem is to find u_0, u_1, v_0 in terms

of f_{-1}, f_{-2}, \dots

$$L^3 = (\partial + f_{-1} \partial^{-1} + f_{-2} \partial^{-2} + f_{-3} \partial^{-3} + \dots) \times$$

$$\times (\partial + f_{-1} \partial^{-1} + f_{-2} \partial^{-2} + f_{-3} \partial^{-3} + \dots) =$$

$$= \partial^2 + f_{-1} + f_{-2} \partial^{-1} + f_{-3} \partial^{-2} +$$

$$+ \partial f_{-1} \partial^{-1} + f_{-1} \partial^{-1} f_{-1} \partial^{-1} + f_{-2} \partial^{-2} f_{-1} \partial^{-1} +$$

$$\begin{aligned}
 & + \rho_{-2} \rho_{-2}^{-2} + \rho_{-1} \rho_{-2}^{-1} \rho_{-2}^{-2} + \rho_{-2} \rho_{-2}^{-2} \rho_{-1} \rho_{-2}^{-1} + \dots + \\
 & + \rho_{-3} \rho_{-3}^{-3} + \dots
 \end{aligned}$$

$$\left\{ \begin{aligned}
 u_0 &= 2\rho_{-1} \\
 u_{-1} &= 2\rho_{-2} + \rho_{-1}
 \end{aligned} \right.$$

$$(\rho + \rho_{-1} \rho_{-1}^{-1}) (\rho_{-2} + u_0 + u_{-1} \rho_{-1}^{-1}) =$$

$$\begin{aligned}
 & = \rho_{-2} + \rho_{-1} \rho_{-1}^{-1} + u_0 \rho + \rho_{-1} \rho_{-1}^{-1} + u_{-1} \rho_{-1}^{-1} + u_{-1} \rho_{-1}^{-1} \rho_{-1}^{-1} \\
 & = \rho_{-2} + \rho_{-1} \rho_{-1}^{-1} + \rho_{-1} \rho_{-1}^{-1} + \rho_{-1} \rho_{-1}^{-1} + \rho_{-1} \rho_{-1}^{-1} + \rho_{-1} \rho_{-1}^{-1} + \rho_{-1} \rho_{-1}^{-1}
 \end{aligned}$$

$$\cancel{F_{-1} \rho_{-1}^{-1}} + F_{-1} \rho_{-1}^{-1} = F_{-1} \rho_{-1}^{-1} + \rho_{-1} \rho_{-1}^{-1}$$

$$F_{-1} \rho_{-1}^{-1} + \rho_{-1} \rho_{-1}^{-1} = \rho_{-1} \rho_{-1}^{-1} + \rho_{-1} \rho_{-1}^{-1}$$

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Finally

$$(L^2)_+ = 2^2 + N_0$$

$$N_0 = 2f - 1$$

$$(L^3)_+ = 2^3 + N_1 + 2 + N_0$$

$$N_1 = 3f - 1$$

$$N_0 = 2f - 2 + 3f - 1$$