Sato theory

(1) Binomial coefficients

$$\binom{n}{j} = \frac{n(n-1)\dots(n-j+1)}{j(j-1)\dots 1} = \frac{n(n-1)\dots(n-j+1)}{j!}$$

$$\binom{n}{1} = n \qquad \binom{-n}{1} = -n$$

$$\binom{n}{2} = \frac{n(n-1)}{1 \cdot 2} \qquad \binom{-n}{2} = + \frac{n(n+1)}{2}$$

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{3!} \; ; \; \binom{-n}{3} = - \frac{n(n+1)(n+2)}{6}$$

$$\binom{-n}{4} = \frac{n(n+1)(n+2)(n+3)}{24}$$

$$\binom{-n}{5} = - \frac{n(n+1)(n+2)(n+3)(n+4)}{}$$

Thus we get

$$\binom{-1}{1} = -1 \qquad \binom{-1}{2} = 1 \qquad \binom{-1}{3} = -1 \qquad \binom{-1}{4} = 1$$

$$\binom{-2}{1} = -2 \qquad \binom{-2}{2} = 3 \qquad \binom{-2}{3} = -4 \qquad \binom{-2}{4} = 5$$

$$\binom{-3}{1} = -3 \qquad \binom{-3}{2} = 6 \qquad \binom{-3}{3} = -10 \qquad \binom{-3}{4} = 15$$

$$\binom{-4}{1} = -4 \qquad \binom{-4}{2} = 10 \qquad \binom{-4}{3} = -20 \qquad \binom{-4}{4} = 35$$

(2)

$$\partial^{-1} u = u\partial^{-1} - \partial u \partial^{-2} + \partial^2 u \partial^{-3} - \partial^3 u \partial^{-4} + \partial^4 u \partial^{-5}$$

$$\partial^{-2} u = u\partial^{-2} - 2\partial u \partial^{-3} + 3\partial^2 u \partial^{-4} - 4\partial^3 u \partial^{-5}$$

$$\partial^{-3} u = u\partial^{-3} - 3\partial u \partial^{-4} + 6\partial^2 u \partial^{-5}$$

$$\partial^{-4} u = u\partial^{-4} - 4\partial u \partial^{-5}$$

② Let L is a formal operator

$$L = \partial + f_{-1}\partial^{-1} + f_{-2}\partial^{-2} + f_{-3}\partial^{-3} + \dots$$

$$L^2 = \partial^2 + u_0 + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \dots$$

$$L^3 = \partial^3 + V_1\partial + V_0 + V_{-1}\partial^{-1} + \dots$$

The problem is to find u_0, V_1, V_0 in terms of $f_{-1}, f_{-2} \dots$

$$L^2 = (\partial + f_{-1}\partial^{-1} + f_{-2}\partial^{-2} + f_{-3}\partial^{-3} \dots) \times$$

$$\times (\partial + f_{-1}\partial^{-1} + f_{-2}\partial^{-2} + f_{-3}\partial^{-3} \dots) =$$

$$= \partial^2 + f_{-1} + f_{-2}\partial^{-1} + f_{-3}\partial^{-2} +$$

$$+ \partial f_{-1}\partial^{-1} + f_{-1}\partial^{-1} f_{-1}\partial^{-1} + f_{-2}\partial^{-2} f_{-1}\partial^{-1} +$$

$$+ \partial f_{-2} \partial^{-2} + f_1 \partial^{-1} f_2 \partial^{-2} + f_{-2} \partial^{-2} f_{-1} \partial^{-1} + \dots +$$

$$+ \partial f_{-3} \partial^{-3} + \dots$$

$$\begin{cases} u_0 = 2 f_{-1} \\ u_{-1} = 2 f_{-2} + \partial f_{-1} \end{cases}$$

$$(\partial + f_{-1} \partial^{-1})(\partial^2 + u_0 + u_{-1} \partial^{-1}) =$$

$$= \partial^3 + f_{-1} \partial + u_0 \partial + \partial u_0 + f_{-1} \partial + v_{-1} \partial^{-1} + u_{-1}$$

$$\Rightarrow \partial \quad (L^3)_+ = \partial^3 + v_1 \partial + v_0$$

$$v_1 = u_0 + f_{-1} = 3 f_{-1} +$$

$$v_0 = \partial u_0 + u_{-1} = 2 f_{-2} + 3 \partial f_{-1}$$

Finally

$$(L^2)_+ = \partial^2 + u_0 \qquad u_0 = 2f_{-1}$$

$$(L^3)_+ = \partial^3 + v_1 \partial + v_0$$

$$v_1 = 3 f_{-1}$$

$$v_0 = 2f_{-2} + 3\partial f_{-1}$$