Lecthere 1

Hamiltonian systems

Let $\Omega \in R^{2N}$ be a domain in the eucledean space of dimension $2N$. One can introduce coordinates in Ω — generilized momenta $p_1 \dots p_N$ (thereafter simple momenta) and generalized coordqnats $q_1 \dots q_N$ (thereafter simple coordinates). Let $H = H(p_1 \dots p_N, q_1 \dots q_N, t)$ is a real function ~~dif~~ defined on Ω. One can make p_i, q_i functions on time and introdue the following dynamical system

$$\dot{q}_n = \frac{\partial H}{\partial p_n} \qquad \dot{p}_n = -\frac{\partial H}{\partial q_n} \qquad n = 1, \dots, N \tag{1.1}$$

Here $\dot{f} = \frac{d}{dt} f$ — time derivative

Let us calculate the time derivative of H

$$\frac{dH}{dt} = \sum_{n=1}^{N}\left(\frac{\partial H}{\partial p_n}\dot{p}_n + \frac{\partial H}{\partial q_n}\dot{q}_n\right) + \frac{\partial H}{\partial t} \qquad (1.2)$$

In virtue of (1.1), the expression in brackets is cancelled and

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \qquad (1.3)$$

The system (1.1) is called Hamiltonian system, function H is the Hamiltonian. If $\frac{\partial H}{\partial t} = 0$, the Hamiltonian is the motion constant.

Suppose $I = I(p_i, q_i)$ is some function on Ω. In virtue of (1.1) $I = I(t)$, I is a function on time. ~~What are condit~~ If $\frac{dI}{dt} = 0$ I is the constant of motion. What are conditions for this?

$$\frac{dI}{dt} = \sum_{n=1}^{N} \left(\frac{\partial I}{\partial q_n} \dot{q}_n + \frac{\partial I}{\partial p_n} \dot{p}_n \right) = \sum_{n=1}^{N} \left(\frac{\partial I}{\partial q_n} \frac{\partial H}{\partial p_n} - \frac{\partial I}{\partial p_n} \frac{\partial H}{\partial q_n} \right) = \quad (1.4)$$

$$= 0$$

Let $A = A(p_1, \ldots p_n, q_1 \ldots q_n)$, $B = B(p_1, \ldots p_n, q_1 \ldots q_n)$ two functions on Ω. The third function C is the Poisson brackets of A, B

$C = \{A, B\}$ as follows

$$\{A, B\} = \sum_{n=1}^{\infty} \left(\frac{\partial A}{\partial q_n} \frac{\partial B}{\partial p_n} - \frac{\partial A}{\partial p_n} \frac{\partial B}{\partial q_n} \right) \quad (1.5)$$

Basic properties of Poisson brackets

(1) Poisson bracket is skew-symmetric operation

$$\{A, B\} = -\{B, A\} \quad (1.6)$$

Poisson brackets obey the Jacoby identity

$$\{A\{B,C\}\} + \{C,\{A,B\}\} + \{A\{B,C\}\} = 0 \qquad (1.7)$$

Proof - straightforward

According to (1.5)

$$\frac{dI}{dt} = \{I,H\} \qquad (1.8)$$

Hence I is an integral of motion, if and only if

$$\{I,H\} = 0 \qquad (1.9)$$

<u>Theorem</u>

Let I_1, I_2 are integrals of motion. Then $I_3 = \{I_1, I_2\}$ is the integral of motion either

In virtue of the Jacoby identity (1.7)

$$\{I_3, H\} = \{\{I_1, I_2\}, H\} = -\{H\{I_1, I_2\}\}$$

$$\{H\{I_1, I_2\}\} + \{I_1\{H, I_2\}\} + \{I_2, \{I_1, H\}\} = 0$$

Equation (1.9) means that the motion integrals of any Hamiltonian system compose a Lie Algebra.

Example

Consider ~~mot~~ motion of a particle in a radially symmetric 3-D force field

The motion conserves three components of the angular momenty

$$M_1 = P_2 x_3 - P_3 x_2 \qquad M_2 = P_3 x_1 - P_1 x_3 \qquad M_3 = P_1 x_2 - P_2 x_3$$

The Poisson bracket are

$$\{M_1, M_2\} = M_3 \qquad \{M_3, M_1\} = M_2$$

$$\{M_2, M_3\} = M_1$$

This this the algebra of vector fields in R^3 with respect to operation of taking vector product.

If $\{I_1, I_2\} = 0$

then integrals I_1, I_2 are in involution or commute. In a general case the Lie algebra of the Hamiltonian system L contain some commutative subobjects $\tilde{L} \in L$

Commuting integrals ~~are~~ all (including the Hamiltonian) belong to $\tilde{L}$

Algebra $\tilde{L}$ is a linear space of dimension m. If $m = N$, the system is called <u>integrable</u>, and the following statement holds

<u>Theorem of integrability</u> (Liuville - Arnold)

An integrable system can be integrated explicity

(We explain the exact meaning of this term later on).

Examples of integrale system

1. Suppose $H = \sum_{n=1}^{N} H_n$

$$H_n = H_n(p_n, q_n)$$

Then all particular Hamitonians are

motion constants which, obviously commute

In particulary if

$$H_n = \frac{1}{2}\left(\frac{p_n^2}{m_n} + k_n x_n^2\right)$$ – this is the system

of linear oscillators with frequencies

$$\omega_n^2 = \frac{k_n}{m_n}$$

~~Ex~~ Another example is mentioned above motion of a particle in ~~the~~ the radially symmetric field. In this case $\tilde{L}$ ~~com~~ is composed ~~he~~ by three elements

H, M_i, M^2 $\qquad M^2 = M_1^2 + M_2^2 + M_3^2$

i – is any integer