Lecture 10

Scattering in the Schrödinger equation

We start with equation:

$$\frac{d^2}{dx^2}\Psi + k^2\Psi = u(x)\Psi \qquad -\infty < k < \infty \tag{1.1}$$

$u(x)$-real function satisfying the condition

$$\int_{-\infty}^{\infty} (1+|x|)|u(x)|dx < \infty \tag{1.2}$$

$k = k_n$ is eigenvalue if the solution f_n of equation (1.1) tends to zero at $|x| \to \infty$. It is well known that this solution is unique. Indeed, if Ψ_1, Ψ_2 are two solutions of (1.1) then

$$\{\Psi_1, \Psi_2\} = const = C \tag{1.3}$$

Here $\{\Psi_1, \Psi_2\} = \Psi_{1x}\Psi_2 - Psi_{2x}\Psi_1$ -wronskian of functions Ψ_1, Ψ_2. If Ψ_1, Ψ_2-eigenfunctions, they tend to zero at $|x| \to \infty$, hence $C = 0$ and Ψ_1, Ψ_2 are proportional to each other.

Eigenvalue k_n must be pure imaginary. Indeed, if k_n is complex

$$
\begin{aligned}
\frac{d^2 f_n}{dx^2} + k_n^2 f_n &= uf \\
\frac{d^2 \bar{f}_n}{dx^2} + k_n^2 \bar{f}_n &= u\bar{f}
\end{aligned}
\tag{1.4}
$$

¿From (1.4) one gets

$$
\frac{d}{dx}\{\dot{f}_n, \bar{f}_n\} = (\bar{k}_n^2 - k_n^2)|f_n|^2 \tag{1.5}
$$

after integrating by x one obtains

$$
\bar{k}_n^2 = k_n^2
$$

Apparently $\{f_n, \bar{f}_n\} = 0$, and eigenfunction F can be made real. Let us introduce Jost functions Ψ, Φ-solutions of equation (1.1), defined by boundary conditions

$$
\begin{array}{ll}
\Psi \longrightarrow e^{ikx} & \Phi \longrightarrow e^{-ikx} \\
x \longrightarrow +\infty & x \longrightarrow -\infty
\end{array}
\tag{1.6}
$$

Jost functions satisfy certain integral equations. One can present Ψ in a form

$$
\Psi = c_1 e^{ikx} + c_2 e^{-ikx}
$$

$c_1(x), c_2(x)$ – functions on x

with additional condition

$$
c_1' e^{ikx} + c_2' e^{-ikx} = 0 \tag{1.7}
$$

Hence

$$
\begin{gathered}
\Psi' = ik(c_1 e^{ikx} - c_2 e^{-ikx} \\
\Psi'' + k^2 \Psi = ik(c_1' e^{ikx} - c_2' e^{-ikx}) = u\Psi
\end{gathered}
\tag{1.8}
$$

Combining (1.7), (1.8), one gets

$$
c_1' = \frac{1}{2ik} u \Psi e^{-ikx} \qquad c_2' = -\frac{1}{2ik} u \Psi e^{ikx} \tag{1.9}
$$

Integrating equation (1.9) we take into account boundary conditions

$$
\begin{aligned}
c_1 &= 1 - \frac{1}{2ik}\int_x^\infty u\Psi e^{-iky}dy \\
c_2 &= \frac{1}{2ik}\int_x^\infty u\Psi e^{iky}dy
\end{aligned}
\tag{1.10}
$$

One can introduce a new function $A = \Psi e^{-ikx} = c_1 + c_2 e^{-2ikx}$. From (1.10 we conclude that B satisfies the integral equation

$$
B(x,k) = 1 - \frac{1}{2ik}\int_x^\infty u(y)(1 - e^{2ik(y-x)})A(k,y)dy \tag{1.11}
$$

The same operation can be performed with function Φ. Now

$$
\begin{aligned}
c_1 &= \frac{1}{2ik}\int_x^\infty u\Phi e^{-iky}dy \\
c_2 &= 1 - \frac{1}{2ik}\int_{-\infty}^x u\Phi e^{iky}dy.
\end{aligned}
\tag{1.12}
$$

Let us denote $B = \Phi e^{ikx}$. This function satisfies the integral equation

$$
A(x,k) = 1 - \frac{1}{2ik}\int_x^\infty u(1 - e^{2ik(x-y)})B(k,y)dy \tag{1.13}
$$

Suppose now that $k = \xi + i\eta$, $\eta > 0$

$$
|e^{2ik(y-x)}| = e^{-2\eta(y-x)}
$$

In (1.2) $y > x$ and this exponent tends to zero as $y \to \infty$. In (1.13) $|e^{2ik(x-y)}| = e^{-2\eta(x-y)}$. As far as $y < x$, this exponent also tends to zero $\eta \to \infty$.

Hence both functions A, B could be analytically continued to the upper-plane. They have these asymptotic expansions

$$
\begin{aligned}
B &\to 1 - \frac{1}{2ik}\int_x^\infty u(y)dy \\
A &\to 1 - \frac{1}{2ik}\int_{-\infty}^x u(y)dy
\end{aligned}
\qquad k \to \infty \quad Imk > 0 \tag{1.14}
$$

and

$$\Psi \rightarrow e^{ikx}\left(1-\frac{1}{2ik}\int_x^\infty u(y)dy\right) \qquad \Phi \rightarrow e^{-ikx}\left(1-\frac{1}{2ik}\int_{-\infty}^x u(y)dy\right)$$

Let $k = i\aleph_n$. Then

$$\begin{array}{lll} \Psi|_{k=i\aleph_n} & \rightarrow & e^{-\aleph_n x} \qquad x \rightarrow \infty \\ \Phi|_{k=i\aleph_n} & \rightarrow & e^{\aleph_n x} \qquad x- \rightarrow \infty \end{array}$$

They present the same eigenfunction f_n and can differ only on some factor. Suppose that f_n is designed by asymptotic

$$\begin{array}{lll} f_n & \rightarrow & e^{\aleph_n x} \qquad x \rightarrow -\infty \\ f_n & \rightarrow & b_n e^{\aleph_n x} \qquad x \rightarrow \infty \end{array} \tag{1.15}$$

Hence

$$f_n = \Phi|_{k=i\aleph_n} = b_n \Phi|_{k=i\aleph_n}. \tag{1.16}$$

In this point Ψ and Φ are proportional to each other.

$\bar{\Psi}(k,x) = \Psi(-k,x)$ and $\bar{\Phi}(k,x) = \Phi(-k,x)$ also are solutions of equation (1.1). Apparently, they are analytic in lower half-plane. Solutions $\Psi, \bar{\Psi}$ comprise a fundamental system. Then, one can put

$$\begin{aligned} \Phi(k,x) &= a(k)\Psi(-k,x) + b(k)\Psi(k,x) \\ Phi(-k,x) &= b(-k)\Psi(-k,x) + a(-k)\Psi(k,x) \end{aligned} \tag{1.17}$$

Apparently

$$a(-k) = \bar{a}(k) \qquad b(-k) = \bar{k}(k) \tag{1.18}$$

Note that

$$\{\Psi(k), \Psi(-k)\} = 2ik \qquad \{\Phi(k), \Phi(-k)\} = -2ik \tag{1.19}$$

Calculating $\{\Phi(k), \Phi(-k)\}$ by the use of (1.17) one finds

$$|a(k)|^2 - |b(k)|^2 = 1. \tag{1.20}$$

We will call $\begin{vmatrix} a & b \\ \bar{b} & \bar{a} \end{vmatrix}$-monodromy matrix, according to (1.20) this matrix is unimodular.

Now from (1.17), (1.19) we get

$$a(k) = \frac{1}{2ik}\{\Psi, \Phi\} \qquad b(k) = \frac{1}{2ik}\{\bar{\Psi}, \Phi\} \tag{1.21}$$

Hence $a(k)$ is analytic in the upper half-plane. By plugging (1.16) into (1.21) one gets

$$a \to \frac{1}{2ik}\left\{ik\left(\left(1 - \frac{1}{2ik}\int_{-\infty}^{x} u(y)dy + \dots\right) + \left(1 - \frac{1}{2ik}\int_{x}^{\infty} u(y)dy\right)\right\}$$
$$= 1 - \frac{1}{4k}\int_{-\infty}^{\infty} u(y)dy + \dots (1.22)$$

The scattering amplitude $c(k)$ is defined as follows

$$c(k) = \frac{a(k)}{b(k)}.$$

Also we define $d(k) = \frac{1}{a(k)}$-amplitude of penetration through the potential barrier. From (1.20) we obtain

$$|c(k)|^2 + |d(k)|^2 = 1 \tag{1.23}$$

This is the "unitary condition": By definition the potential $u(x)$ is reflectionless if $r(k) \equiv 0$.

In this case $a(k)$ can be found explicitly from the conditions $|a(k)| = 1$ for real k, $a(-k) = \bar{a}(k)$ $\quad a(k) \to 1 \quad k \to \infty$; $a(k)$-analytic in the upper half-plane.

If $a(k)$ has no zeros in upper half-plane then $a(k) \equiv 1$. In virtue of condition $a(-k) = \bar{a}(k)$ all zeros are posed on the imaginary axis. Apparently they are exact eigenvalues $\aleph_n$. $a(k)$ can be presented as the product

$$a(k) = \prod_{m=1}^{n} \frac{k - i\aleph_n}{k + i\aleph_n} \tag{1.24}$$

For reflectionless potential function

$$Y_u\cancel{(k,x)} = \frac{\cancel{B}A(k,x)}{a(k)} = \cancel{B}b(-k,x) \tag{1.25}$$