Math 488-588

Lecture 11.

$$\psi_{xx} + (k^2 - u)\psi = 0 \qquad (1)$$

Jost functions

$\varphi \to e^{-ikx} \quad x \to -\infty$ $\qquad \psi \to e^{ikx} \quad x \to +\infty$

φ, ψ are analytic in the upper half-plane

$$\varphi = a(k)\,\psi(-k) + b(k)\,\psi(k)$$

$a(k) = \frac{1}{2ik}\{\psi, \varphi\} = -\frac{1}{2ik}\{\varphi, \psi\}$ – analytic (2)

in the upper half plane

$\varphi = e^{-ikx} A(k,x)$ $\qquad \psi = e^{ikx} B(k)$

$$A(k,x) = a(k) + B(k) \cdot b(k) e^{2ikx}$$

If $\quad k \to k_R + i k_I \qquad k_I > 0$

$$a(k) = \lim_{x \to \infty} A(k,x)$$

$$A_{xx} - 2ik A_x - u A = 0$$

$$A \to 1 \qquad x \to -\infty$$

We introduce

$$A = \exp \int_{-\infty}^{x} \chi(x', k)\, dx'$$

$$\ln a(k) = \int_{-\infty}^{\infty} \chi(x, k)\, dx \qquad (3)$$

$$(4)$$

$$\chi_x + \chi^2 - u = 2ik\chi$$

$$\chi = \sum_{n=1}^{\infty} \frac{\chi_n}{(2ik)^n} \qquad (5)$$

$$\chi_1 = -u$$

$$\chi_{n+1}(x) = \frac{d}{dx} \chi_n(x) + \sum_{l=1}^{n-1} \chi_l \chi_{n-l} \qquad (5)$$

$$\chi_2 = -u_x \qquad (6)$$

$$\chi_3 = -u_{xx} + u^2 \tag{7}$$

$$\chi_4 = -u_{xxx} + 2\frac{d}{dx}u^2 \tag{8}$$

$$\chi_5 = -u^{(4)}_x + \frac{d^2}{dx^2}u^2 + 2u_{xx}u - 2u^3 \tag{9}$$

All χ_n are real

One can separate

$$\chi = \chi_R + i\chi_I \tag{10}$$

From (4) one obtains

$$\chi_{Ix} + 2\chi_I\chi_R - 2k\chi_R = 0 \tag{11}$$

$$\chi_R = \frac{1}{2}\frac{1}{\chi_I - k}\frac{d\chi_I}{dx} = \frac{1}{2}\frac{d}{dx}\ln \chi_I \tag{12}$$

$$\chi_R = \sum_{\ell=1}^{\infty}\frac{\chi_{2\ell}}{(2ik)^{2\ell}} \tag{13}$$

As far as χ_{2l} is a complete derivative

$$\int_{-\infty}^{\infty} \chi_{2l}\, dx = 0$$

$$I_n = \int_{-\infty}^{\infty} \chi_{2n+1}\, dx$$

$$I_1 = -\int_{-\infty}^{\infty} u\, dx = -N$$

$$I_2 = \int u^2 dx = P$$

$$I_3 = -\int (u_x^3 + 2u^3)\, dx = -2H$$

$$H = \int \left(\frac{1}{2} u_x^2 + u^3\right) dx$$

$$\frac{\delta H}{\delta u} = - u_{xx} + 3u^2$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \frac{\delta H}{\delta u}$$

$$\frac{\partial u}{\partial t} - 6 u u_x + u_{xxx} = 0$$

All I_n are motion integrales of the KdV equation

Let us consider pure solitonic case

$$|a(k)|^2 - |b(k)|^2 = 1 \qquad b(k) = 0$$

$$a(k) = \prod_n \frac{k - i\varkappa_n}{k + i\varkappa_n}$$

$$\ln a(k) = \sum_{k} \ln(k - i\varkappa_n) - \ln(k + i\varkappa_n) =$$

$$= \sum \Big[\ln\left(1 - \frac{i\varkappa_n}{k}\right) - \ln\left(1 + \frac{i\varkappa_n}{k}\right) =$$

$$= 2 \sum \frac{1}{2l+1} \left(\frac{-i\varkappa_n}{k} \right)^{2l+1} = 2 \sum \frac{2^{2l+1}}{2l+1} \left(\frac{\varkappa_n}{2ik} \right)^{2l+1} \quad (*)$$

Comparing $(*)$ with (5) one obtains

$$\int_{-\infty}^{\infty} \chi_{2l+1}(x)\,dx = \frac{2^{2l+2}}{2l+1} \sum \varkappa_n^{2l+1}$$

In particular

$$N = \int_{-\infty}^{\infty} u\,dx = -4 \sum \varkappa_n < 0$$ in pure solitonic case

$$\int u^2\,dx = 8 \sum \varkappa_n^3$$

$$H = \int \left(\frac{1}{2} u_x^2 + u^3 \right) dx = -\frac{32}{5} \sum \varkappa_n^5$$

In pure solitonic case

$N < 0 \qquad H < 0$