Lecture 12

In the next two lectures the following basic lemma will play an important role

Basic lemma

Let $f_1(x)$, $f_2(x)$ two solutions of the Schrödinger equation with different values of k

$$f_{1xx} + (k_1^2 - u) f_1 = 0 \qquad (1)$$

$$f_{2xx} + (k_2^2 - u) f_2 = 0$$

Then

$$f_1 f_2 = \frac{1}{k_2^2 - k_1^2} \frac{d}{dx} \{f_1, f_2\} \qquad (2)$$

Recall that the Wronskian

$$\{f_1, f_2\} = f_2 \frac{\partial f_1}{\partial x} - f_1 \frac{\partial f_2}{\partial x} \qquad (3)$$

A proof of this lemma is straightforward

The basic lemma makes possible to calculate Poisson brackets betwenn different elements of scattering data. Let φ_1, φ_2 are two eigenfunctions of the Schrödinger operator

$$\frac{d^2\varphi_1}{\partial x^2} - E_1\varphi_1 = u\varphi_1 \qquad \varphi_{1,2} \to 0$$

at $x \to \pm\infty$

$$\frac{\partial \varphi_2}{\partial x^2} - E_2\varphi_2 = u\varphi_2$$

Let us calculate Poisson bracket

$$\{E_1, E_2\} = \frac{1}{2}\int\left(\frac{\delta E_1}{\delta u}\frac{\partial}{\partial x}\frac{\delta E_2}{\delta u} - \frac{\delta E_2}{\delta u}\frac{\partial}{\partial x}\frac{\delta E_1}{\delta u}\right)dx \quad (4)$$

We start from equation

$$\frac{d^2\varphi}{dx^2} - E\varphi(x) = u(x)\varphi(x) \quad (5)$$

and calculate equation for the variational derivative

$$g(x,z) = \frac{\delta\varphi(x)}{\delta u(z)}.$$ Remember that $\frac{\delta u(x)}{\delta u(z)} = \delta(x-z)$ (6)

It satisfies to equation

$$\frac{d^2 g}{dx^2} - E\, g - u g = \frac{\delta E}{\delta u(z)}\varphi(x) + \delta(x-z)\varphi(x) \quad (7)$$

~~Subtracti~~ ~~Subtracting~~ ~~subtracting~~

Then we multiply (5) to $g(x,z)$ ~~and~~
~~subtract~~ (7) to $\varphi(x)$ and subtract the results

One receives

$$\frac{\delta E}{\delta u(z)} \varphi^2(x) + \delta(x-z)\varphi^2(x) = \varphi(x)\frac{\partial^2 g}{\partial x^2} - g\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial}{\partial x}\{g, \varphi\} \tag{8}$$

Integration of (8) by x gives the result known from the theory of perturbation

$$\frac{\delta E}{\delta(z)} = -\varphi^2(z) \tag{9}$$

Hence

$$\{E_1, E_2\} = \frac{1}{2}\int\left[\varphi_2^2(x)\frac{\partial}{\partial x}\frac{\varphi_1^2}{\partial x} - \varphi_1^2(x)\frac{\partial}{\partial x}\varphi_2^2(x)\right] dx =$$

$$= \int \varphi_1\varphi_2\{\varphi_1,\varphi_2\} = \frac{1}{E_1 - E_2}\int_{-\infty}^{\infty}\frac{d}{dx}\{\varphi_1,\varphi_2\}\cdot\{\varphi_1\varphi_2\} =$$

$$= \frac{1}{E_1 - E_2}\,\frac{1}{2}\int_{-\infty}^{\infty}\frac{d}{dx}\{\varphi_1,\varphi_2\}^2\, dx = 0$$

Finally we obtain

$$[E_1, E_2] = 0. \tag{10}$$

In other words discrete eigenvalues of the Schrödinger equation commute.

Let us return to the Jost function $\Phi(x, k)$ satisfying to equation

$$\frac{d^2\varphi}{dx^2} + k^2\varphi = u(x)\varphi \qquad (11)$$

with boundary condition

$$\varphi \to e^{-ikx} \qquad x \to -\infty$$

~~It~~ We denote

$$G(x, z) = \frac{\delta\varphi(x)}{\delta u(z)} \qquad (12)$$

Apparently $G(x, z) = 0$ at $x < z$. It satisfies to equation

$$\left(\frac{d^2}{dx^2} + k^2\right) G(x, z) = u(x)\, G(x, z) + \delta(x - z)\,\Phi(x) \qquad (13)$$

But $\delta(x - z)\,\varphi(x) = \delta(x - z)\,\varphi(z)$

Hence

$$g(x,z) = \varphi(z)\, K(x,z) \qquad (14)$$

Here $K(x,z)$ the Green function of the Schrödinger ~~equation~~ operator, satisfying to equation

$$\left(\frac{d^2}{dx^2} + k^2\right) K(x,z) = u(x)\, K(x,z) + \delta(x-z) \qquad (15)$$

$$K(x,x) = 0 \qquad \left.\frac{\partial K}{\partial x}\right|_{x=z+\varepsilon} = 1 \qquad (16)$$

For ~~z~~ $x > z$ K is a solution of homogenous Schrodinger equation. This solution must satisfy to condition (16). That is

$$K(x,z) = \frac{1}{2ik}\left[\Psi(x,k)\,\overline{\Psi}(z,k) - \overline{\Psi}(x,k)\,\Psi(z,k)\right] \qquad (17)$$

Hence

$$g(x,z) = \frac{\varphi(z,k)}{2ik\,\cancel{2\pi}}\left\{\Psi(x,k)\,\overline{\Psi}(z,k) - \overline{\Psi}(x,k)\,\Psi(z,k)\right\}$$

$$g(x,z) \to \cancel{\varepsilon}\, \frac{1}{2ik}\left\{-\varphi(z,k)\Psi(z,k)\, e^{-ikx} + \right.$$

$$\left. + \varphi(z,k)\,\overline{\Psi}(z,k)\, e^{ikx}\right\}$$

The other hand

$$g(x,z) \to \frac{\delta a(k)}{\delta u(z)} e^{-ikx} + \frac{\delta b(k)}{\delta u(z)} e^{ikx}$$

We end up with the remarkable result

$$\frac{\delta a(k)}{\delta u(z)} = -\frac{1}{2ik} \varphi(z,k) \psi(z,k)$$

$$\frac{\delta b(k)}{\delta u(z)} = \frac{1}{2ik} \varphi(x,k) \overline{\psi}(z,k)$$

As expected $\frac{\delta a(k)}{\delta u(z)}$ is analytic ~~at~~

on the upper half-plane