Lecture 14

Derivation of the Marchenko equation

Let us return to the Schrödinger equation

$$\psi'' + k^2\psi = u\psi \quad \text{or} \quad -\psi'' + u\psi = E\psi$$

Most functions

$\varphi \to e^{-ikx}$ $\quad x \to -\infty$

$\psi \to e^{ikx}$ $\quad x \to +\infty$

are connected with the relation

$$\varphi(k,x) = a(k,x)\,\psi(x,-k) + b(k,x)\,\psi(x,k) \qquad (1)$$

Functions φ and ψ – are analytic in the upper half-plane. As far

$$a(k) = \frac{1}{2ik}[\varphi, \psi],$$

~~this~~ function also is analytic. In the point of discrete spectrum $a(k)$ has zeros

$$a(i\varkappa_n) = 0$$

If $E_n = -\varkappa_n^2$

$a(i\varkappa) = 0$ and functions Ψ и Φ are proportional to each other

$$\varphi_n = \varphi(i\varkappa_n), \quad \Psi_n = \Psi(i\varkappa_n) \qquad (4)$$

$\varphi_n = c_n \Psi_n$

Now φ_n, Ψ_n are real eigenfunctions, c_n — real coefficients, positive or negative

Eigenfunction φ_n has asymptotics

$$\varphi_n \to e^{\varkappa_n x} \quad x \to -\infty \qquad (5)$$

$$\varphi_n \to c_n e^{-\varkappa_n x} \quad x \to +\infty$$

It has a finite norm

$$\int_{-\infty}^{\infty} \varphi_n^2 \, dx = \|\varphi_n\|^2 \qquad (6)$$

There is an important question how c_n and $\|\varphi_n\|^2$ are connected?

Let us differentiate equation (1) by k

$$\varphi_k'' + k^2 \varphi_k = u\varphi_k - 2k\varphi \qquad (7)$$

Let $y = \{\varphi, \varphi_k\}$ — a Wronskian

From (1), (7) one obtains equation

$$\frac{d}{dx}\{\varphi, \varphi_k\} = -2k\varphi^2 \qquad (8)$$

If $k = i\varkappa_n$, this equation can be integrated

$$-2i\varkappa_n \|\Phi_n\|^2 = \lim_{x\to\infty}\{\varphi_n \varphi_k\} \qquad (9)$$

Indeed $\{\varphi, \varphi_k\} \to 0$ at $x \to -\infty$

We know that

$$\varphi_n \to c_n e^{-\varkappa_n x}$$

Equation (9) means that $\varphi_k\big|_{k=i\varkappa} = \alpha e^{\varkappa_n x}$

Our problem is in determining of α and will

To do this we return to equation (2) and will try to continue it to the upper half-plane.

In a general case $\Psi(k,x)$ can be continued only to the lower half-plane, while $b(k)$ does not admit an analytic continuation from the real axis at all, but we will use a special trick.

We replace in (1) the potential u to the potential and replace in (1)

$$u \to \tilde{u} \qquad \tilde{u} = \begin{cases} u & |x| < L \\ 0 & |x| > L \end{cases} \qquad (11)$$

~~and in the end~~

In the end we will tend $L \to \infty$

For any finite value of L all functions having no singularities $\Phi, \Psi, \overline{\Psi}$ are entire functions on the whole complex plane. The same statement is correct for coefficients $a(k)$, $b(k)$. If L is large enough $a(x)$ ~~has~~ has the same amount of zeros in slightly shifted points ~~$x_n \to x_n$~~

$$x_n(L) \to x_n \quad \text{as } L \to \infty$$

Then we denote

$$b_L(ix_n)$$

The fuction $b_L(k)$ might aquier singularities if ~~as~~ $L \to \infty$ but the value $b_L(ix)$ becomes finite, moreover

$$\lim_{L \to \infty} b_L(i x_n(L)) \to c_n \qquad (12)$$

truncated potential

$$u \to \tilde{u} \qquad \tilde{u} = \begin{cases} u & |x| < L \\ 0 & |x| > L \end{cases}$$

In the end of this procedure we will tend $L \to \infty$

At ~~finite~~ any large, but still finite L all ~~on~~

functions φ, ψ, a, b in (2) are entire functions plane k.

functions on the whole complex

having no singularities has the same amount

If L is large enough, $a(k)$ as in the limiting

zeros on the imaginary axis is the zero which will

case $L = \infty$ Let $\varkappa_n(L)$ ~~Let us differe~~ Let us

turn to $\varkappa_n$ when $L \to \infty$. ~~and~~, make analytic

differentiate relation (2) by k half-plane and put

continuation to the upper study ~~it~~ the result

$k = i\varkappa_n(L)$ and ~~consider~~ ~~Then~~ As far as $a(i\varkappa_n(L)) = 0$

in any point $x > L$.

We obtain the result

$$\varphi_k(x)\Big|_{k = i\varkappa} = a'_k e^{\varkappa_n(L)x} + \frac{d}{dk} b(k) e^{-\varkappa_n(L)x} \qquad (13)$$

Then take in account that in this approximation (for $x > L$)

$$\varphi_n(x) = b(i\varkappa_n(L)) e^{-\varkappa_n(L)x} \qquad (14)$$

The second term in (13) vanishes at $x \to \infty$. Hence we get

$$\lim_{x\to\infty} \{\varphi, \varphi\} = + a'(i\varkappa)\, b(i\varkappa)\, 2\varkappa_n$$

Now we tend $L \to \infty$ and end up with the answer

$$\|\varphi_n\|^2 = \int_{-\infty}^{\infty} \varphi_n^2\, dx = + i\, a'(i\varkappa_n)\, c_n \qquad (15)$$

Equation (15) proves that $i a'(i\varkappa_n)$ is a real number. Let us check this fact for the pure solitonic case. Let the eigenvalues are ordered as follows

$$\varkappa_1 > \varkappa_2 > \dots > \varkappa_N$$

$$a(k) = \prod \frac{k - i\varkappa_n}{k + i\varkappa_n} \qquad (16)$$

$$i a'(i\varkappa_n) = \frac{1}{2\varkappa_n} \prod_{m \neq n} \frac{\varkappa_n - \varkappa_m}{\varkappa_n + \varkappa_m}$$

One can see that $i a'(i\varkappa_1) > 0$ $i a'(i\varkappa_2) < 0$ while $c_1 > 0$ $c_2 > 0$

$i a'(ix_n)$, c_n are affinely intermitted real number, their product is always positive

Now mention that $(a'(ix_n))^2 < 0$ is a negative real number. Then quaitiy

$$\frac{i c_n}{a'(ix)} = - M_n^2 = \frac{1}{(a'(ix_n))^2} \|\varphi_n\|^2 \quad \text{is} \qquad (17)$$

a set of negative real numbers