Lecture 14

The Marchenko equation (continuation)

Remember that

$$\varphi(x,k) = A(x,k)\, e^{-ikx} \qquad (1)$$

$$\psi(x,k) = B(x,k)\, e^{ikx}$$

Functions $A(x,k)$, $B(x,k)$, both analytic in the upper half-plane are connected with relation

$$A(x,k) = a(k)\, B(x,-k) + e^{-2ikx}\, b(k)\, B(x,k) \qquad (2)$$

Dividing both parts of (2) gives

$$\frac{A(x,k)}{a(k)} = B(x,-k) + e^{-2ikx}\, c(k)\, B(x,k) \qquad (3)$$

Here $c(k)$ is the reflection coefficient

In virtue of relation $|a|^2 - |b|^2 = 1$

$$|c|^2 \leq 1 \qquad c(k) \text{ is the reflection coefficient} \qquad (4)$$

Let us introduce the function $f(k, x)$ as follows

$$f = \frac{A(x,k)}{a(k)} \qquad \operatorname{Im} k > 0 \tag{5}$$

$$f = B(-k) \qquad \operatorname{Im} k < 0$$

Function f is analytic on all complex plane except the real axis. Moreover it has simple poles on the positive imaginary axis.

One can denot

$$\cancel{\frac{A(x,k+i\varepsilon)}{a(k+i\varepsilon)}} \lim_{\varepsilon \to 0} \frac{A(x,k+i\varepsilon)}{a(k+i\varepsilon)} = f^{+}(k,x) \tag{6}$$

In the same way

$$B(-k+i\varepsilon) = f^{-}(k,x) \tag{7}$$

$$B(x,k) = f^{+}(-k,x)$$

Then relation

$$\frac{A(x,k)}{a(k)} - B(x,-k) = e^{-2ikx} c(k)\, B(x,k) \qquad (8)$$

Can be rewritten as follows

Function χ satisfies to the Riemann-Hilbert problem

$$\chi^{+}(k) - \chi^{-}(k) = e^{-2ikx} c(k)\, \chi^{+}(-k,x) \qquad (9)$$

~~...~~ Function χ is a solution of the following equation

$$\frac{\partial^2 \chi}{\partial x^2} + 2ik \frac{\partial \chi}{\partial x} = u\chi \qquad (10)$$

It has asymptotic expansion at infinity

$$\chi = 1 + \frac{\chi_0}{ik} + \dots \qquad (11)$$

Apparentey

$$u = 2 \frac{\partial \chi_0}{\partial x} \qquad (12)$$

now remember that

$$B(k) \to 1 - \frac{1}{2ik} \int_x^\infty u(y)\,dy \qquad (13)$$

Let us calculate the following integral

$$K(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (B(k) - 1)\, e^{ik(x-y)}\, dk$$

$B(k) - 1 \to 0$ at $k \to \infty$ and analytic in the upper half-plane

Then if $x > y$ the integral can be closed ~~through~~ over the upper half plane, hence

$$K(x,y) \equiv 0 \qquad \text{if } x > y \qquad (14)$$

(This is the Paley–Wiener theorem)

Returning to the Jost function ψ we realize that it has following famous triangle representation

$$\psi(x,k) = e^{ikx} + \int_x^\infty K(x,y)\, e^{iky}\, dy \qquad (15)$$

Putting in (15) $k = i\varkappa_n$ we find that the eigenfunction

$$\psi_n = \psi(x, i\varkappa_n) = e^{-\varkappa_n x} + \int_x^\infty K(x,y)\, e^{-\varkappa_n y}\, dy \qquad (16)$$

Hence

$$\varphi_n = c_n \left(e^{-\varkappa_n x} + \int_x^\infty K(x,y)\, e^{-\varkappa_n y}\, dy \right) \qquad (17)$$

Notice also that

$$\psi(x,-k) = e^{-ikx} + \int_x^\infty K(x,y)\, e^{-iky}\, dy \qquad (18)$$

The basic relation between φ and ψ can be rewritten as follows

$$\frac{\varphi(x,k)}{a(k)} - e^{-ikx} = \int_x^\infty K(x,y)\, e^{-iky} dy + c(k) \left[e^{ikx} + \int_x^\infty K(x,y)\, e^{iky} dy \right] \qquad (19)$$

Let us multyply this equation to $\frac{1}{2\pi} e^{ikz}$ $z > x$ and integrate by k.

As far as $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(z-x)}\, dx = \delta(x-z)$

the right hand of integral is easily taken

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\varphi(x,k)}{a(k)} - e^{-ikx}\right] e^{ikz} = K(x,z) +$$

$$+ \tilde{F}(x+z) + \int_x^{\infty} \cancel{\tilde{F}} \, K(x,s)\,\tilde{F}(s+z)\,ds$$

$\cancel{F(z) = \frac{1}{2\pi}\int}$

The ~~e~~ left part ~~can be taken~~ is the sum of residues

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\varphi(x,k)}{a(k)} - e^{-ikx}\right] e^{ikz}\,dk = i\sum \frac{\varphi_n(x)}{a'(i\varkappa_n)}$$

Now we use ~~eta~~ equation (17) and equation (17) from the previous lecture. We end up with the famous Marchenko equation

$$K(x,z) + F(x+z) + \int_x^{\infty} K(x,s)\,F(s+z)\,dz = 0$$

Here

$$F(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(k)\, e^{ik\xi} + \sum M_n^2\, e^{-\varkappa_n \xi}$$

This equation is completely equivalent to the Riemann-Hilbert problem presented in the lecture 6

$$\frac{\partial \psi}{\partial \bar{k}} = R(k)\, \psi(-k)\, e^{-2ikx}$$

Where the dressing function R consist of two parts

$$R(k) = \sum M_n^2\, \delta(k - i\varkappa_n) + c(k_R)\, \delta^{*}(k_I)\, \theta(k_I)$$

Here we denote

$$k = k_R + i k_I$$

$$\theta(k_I) = \begin{cases} 1 & k_I > 0 \\ 0 & k_I < 0 \end{cases}$$