Lecture 15

The NLS equation

Let Ψ is a complex valued $N \times N$ matrix satisfying to the system of compatible equations

$$\Psi_x = \hat{U} \Psi \qquad (1)$$

$$i \Psi_t = \hat{V} \Psi \qquad (2)$$

Here $\hat{U} = \hat{U}(x, t, \lambda)$ $\hat{V} = V(x, t, \lambda)$ — rational functions of λ.

The compatibility conditions for system (1.2) is

$$i \hat{U}_t - \hat{V}_x + [\hat{U}, \hat{V}] = 0 \qquad (3)$$

Suppose

$$\hat{U} = I\lambda + u \qquad (4)$$

$$\hat{V} = \lambda \hat{U} + v = I\lambda^2 + u\lambda + v$$

u, v do not depend on λ.

Equation (3) reads

$$v_x = [I, u] \qquad (5)$$

$$u_t - v_x + [u, v] = 0 \qquad (6)$$

Thereafter we assume $N = 2$

$$I = \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad u = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}$$

Notice that now

$$\hat{V} = \lambda \hat{U} +$$

From equation (6) one get

$$V = \begin{bmatrix} \mathcal{A} & p \\ q & -\mathcal{A} \end{bmatrix} \qquad \operatorname{tr} V = 0$$

A, B – some unknown functions

$$W = \begin{bmatrix} B & w_{12} \\ w_{21} & -B \end{bmatrix} \qquad (9)$$

Notice that

$$[u, v] = -2\mathcal{A} \begin{bmatrix} 0 & p \\ -q & 0 \end{bmatrix}$$

From equation (7) one obtains

$A_x = 0$. One can put $\mathcal{A} = 0$ $W = U$

Now

$$\hat{V} = \lambda \hat{U} + W$$

Equation (7) is

$$[I, w] = u_x$$

Hence

$$w_{12} = \frac{1}{2} p_x \qquad w_{21} = -\frac{1}{2} q_x$$

Then

$$[u, w] = -2B \begin{bmatrix} 0 & p \\ -q & 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}(pq)_x & 0 \\ 0 & \frac{1}{2}(pq)_x \end{bmatrix}$$

Hence

$$-B_x - \frac{1}{2}(pq)_x = 0 \qquad B = -\frac{1}{2} pq$$

Equation (8) is reduced to two nonlinear PDE

$$i p_t - \frac{1}{2} p_{xx} + p^2 q = 0 \qquad (10)$$

$$i q_t + \frac{1}{2} q_{xx} - p q^2 = 0 \qquad (11)$$

Equations (10) (11) admit following basic reductions

1. $q = \bar{p}$

Now p obeys the "defocusing Nonlinear Schrödinger Equation" (defocusing NLSE)

$$i p_t - \frac{1}{2} p_{xx} + |p|^2 p = 0 \qquad (12)$$

2. $q = -\bar{p}$

In this case p obeys the focusing NLSE

$$i p_t - \frac{1}{2} p_{xx} - |p|^2 p = 0 \qquad (13)$$

Both equations are Hamiltonian

They can be written in the form

$$i p_t + \frac{\delta H}{\delta \bar{p}} = 0$$

In the defocusing case

$$H = \frac{1}{2}\int |p_x|^2 dx + \frac{1}{2}\int |p|^4 dx \qquad (14)$$

In the focusing case

$$H = \frac{1}{2}\int |p_x|^2 dx - \frac{1}{2}\int |p|^4 dx \qquad (15)$$

The focusing equation (13) has a rich family of solitonic solutions. The simpliest ones are resting solitons

$$p = \varphi(x)\, e^{-\frac{i}{2}\lambda^2 t}$$

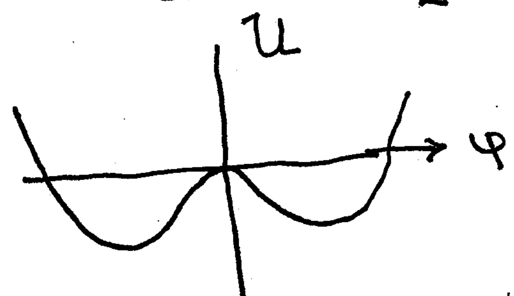
$\varphi(x)$ satisfies the equation

$$\varphi'' - \lambda^2\varphi + 2\varphi^3 = 0 \qquad (16)$$

This equation (16) describe motion of a particle inside the potential well

$$\varphi'' + \frac{\partial U}{\partial \varphi} = 0 \qquad (17)$$

$$U = -\frac{1}{2}\lambda^2\varphi^2 + \frac{1}{2}\varphi^4$$



Equation (17) has the motion integral

$$\frac{1}{2}\varphi_x^2 + U = E$$

Let us put $E = 0$

Equation

$$\varphi_x^2 - \lambda^2 \varphi + \varphi^4 = 0$$

has following exact solution

$$\varphi = \frac{\lambda}{\cosh \lambda x}$$

This solution can be generalized

$$P = \frac{\lambda}{\cosh \lambda (x - x_0)} \, e^{i\xi}$$

x_0, ξ — arbitrary real number

A more general solutions can be found in the form

$$\Psi = e^{i(ax + bt)} \varphi(x - vt) \qquad y = x - vt$$

The real function $\varphi(y)$ satisfies to

$$\left(-\mathcal{E}+\frac{1}{2}a^2\right)\varphi - i(V+a)\varphi_y = \frac{\lambda}{2}\varphi_{yy} + \varphi^3$$

Let us choose

$$a = -V \qquad \frac{1}{2}a^2 - \mathcal{E} = \frac{1}{2}\lambda^2 \quad \Big| \quad \mathcal{E} = \frac{1}{2}\left(V^2-\lambda^2\right)$$

Then the expression

$$\Psi = \lambda \frac{e^{-iVx + \frac{i}{2}(V^2-\lambda^2)t + i\xi}}{\cosh \lambda(x - Vt - x_0)}$$

is a general solitonic solution of the focusing NLSE. It depends on four arbitrary parameters λ, V, x_0, ξ. A soliton can move with an arbitrary velocity in both directions