~~Lecture~~ 16 (16)

Condensate and its stability

Both focusing and defocusing equation can be written in the unified form

$$i\Psi_t = \frac{1}{2}\Psi_{xx} + s|\Psi|^2\Psi \qquad (1)$$

$$s = \pm 1.$$

Let us separate the amplitude and the phase

$$\Psi = A e^{-i\varphi}$$

$$\Psi_t = (A_t - iA\varphi_t) e^{-i\varphi}$$

$$\Psi_x = (A_x - iA\varphi_x) e^{-i\varphi_x}$$

$$\Psi_{xx} = A_{xx} - 2iA\varphi_x - iA\varphi_{xx} - A\varphi_x^2$$

how vector equation (1) is splitted to two scalar equations

$$A_t + A_x \varphi_x + \frac{1}{2} A \varphi_{xx} = 0 \qquad (2)$$

$$A\left(\varphi_t + \frac{1}{2}\varphi_x^2\right) - sA^3 - \frac{1}{2} A_{xx} = 0 \qquad (3)$$

One can introduce

$$\rho = A^2$$

Equation (2) takes form of the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \rho \varphi_x = 0 \qquad (4)$$

Equation (3) becomes Bernoulli equation with additional dispersive term

$$\varphi_t + \frac{1}{2}\varphi_x^2 - s\rho - \frac{1}{2\sqrt{\rho}} \left(\sqrt{\rho}\right)_{xx} = 0 \qquad (5)$$

Equations (4) (5) are Hamiltonian

$$\frac{\partial \rho}{\partial t} = \frac{\delta H}{\delta \varphi}$$

$$\frac{\partial \varphi}{\partial t} = -\frac{\delta H}{\delta \rho}$$

$$H = \frac{1}{2}\int (\sqrt{\rho})_x^2\, dx - \frac{1}{2}s\int \rho^2\, dx + \frac{1}{2}\int \rho\, \varphi_x^2\, dx \qquad (6)$$

The simpleest of equation (1) is the condensate

$\Psi = A_0 e^{-is|A_0|^2 t}$ A_0 – an arbitrary complex number $A_0 = |A_0| e^{i\xi}$. Thereafter we will assume A_0 ~~real~~ to be real in the case

$$\varphi = s A_0^2$$

One can replace $\varphi \to s A_0^2 t \varphi$

Then equation (3) transforms to

$$A\left(\varphi_t + \frac{1}{2}\varphi_x^2\right) - sA(A^2 - A_0^2) - \frac{1}{2}A_{xx} = 0 \qquad (6)$$

Let us put

$$A = A_0(1+u) \qquad u \ll 1$$

and consider $\varphi_x \ll 1$. Equations (3, 6) ~~The system~~ ~~is~~ are simplified to the linear system

$$u_t + \frac{1}{2}\varphi_{xx} = 0$$

$$\varphi_t - 2sA_0^2 u - \frac{1}{2}u_{xx} = 0$$

Assuming

$u, \varphi \simeq e^{-i\omega t + ipx}$ one easily find

the dispersion relation

$$\omega^2 = \left(-sA_0^2 + \frac{1}{4}p^2\right)p^2 \qquad (7)$$

This is the famous Bogolyubov formula

In the defocusing case

$s = -1$

$$\omega^2 = \left(A_0^2 + \frac{1}{4}p^2\right)p^2 \qquad (8)$$

at $p \to \infty$ $\qquad \omega \to \frac{1}{2}p^2$

at $p \to 0$ $\qquad \omega \to A_0 p$

In the focusing case $s = 1$

$$\omega^2 = \left(-A_0^2 + \frac{1}{4}p^2\right)p^2$$

The condensate is unstable if $p^2 < 4A_0^2$

This is the modulational instability. The growth-rate maximum is achieved at

$$p^2 = 2A_0^2$$

In this point

$$\operatorname{Im} \omega = A_0^2$$

Equation (1) describes the one-dimensional Bose-gas in the classical wave limit

The interaction between particles is extremely short-scale. If $s=-1$, the interaction is repelling in the Bose-condensate is stable

If $s=1$, the interation is attraction, and the Bose-condensate is unstable.

I recommend to my class the following paper

" Superregular solitonic solutions: a novel scenario for the nonlinear stage of modulational instability"
A.A. Gelash, and V. E. Zakharov
Nonlinearity 27 (2014) R1-R33