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~~Lecture~~ Lecture 17

Dark solitons

In this lecture we study solitons in the defocusing NLSE. They exist only on a nontrivial condensate. We start with the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \rho \varphi_x = 0 \qquad (1)$$

and will assume that $\rho = \rho(x - Vt)$

$$\varphi_x = \varphi_x(x - Vt) \qquad \varphi_t = -V\varphi_x.$$

$$\frac{\partial \rho}{\partial t} = -V \frac{\partial \rho}{\partial x} \qquad \rho \to \rho_0 \quad \text{at } |x| \to \pm\infty$$

Thereafter we denote $\varphi_x = u$

Now we assume $u \to 0$ at $x \to \pm\infty$

The continuity equation gives

$$\frac{\partial}{\partial x}\rho u = V\frac{\partial \rho}{\partial x}$$

or

$$\rho u = V(\rho - \rho_0) \qquad u = V\left(1 - \frac{\rho_0}{\rho}\right) = V\left(1 - \frac{A_0^2}{A^2}\right)$$

The Bernoully equation gives

$$A\left(-Vu + \frac{1}{2}u^2\right) + A\left(A^2 - A_0^2\right) - \frac{1}{2}A_{xx} = 0$$

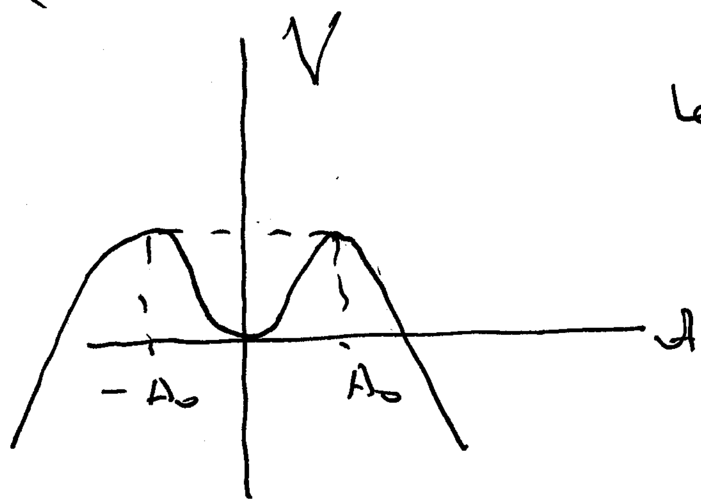
After simple calculations we end up with equation

$$A_{xx} - \lambda A^3 + \left(V^2 + 2A_0^2\right)A - \frac{V^2 A_0^4}{A^3} = 0$$

This equation can be rewritten as follows

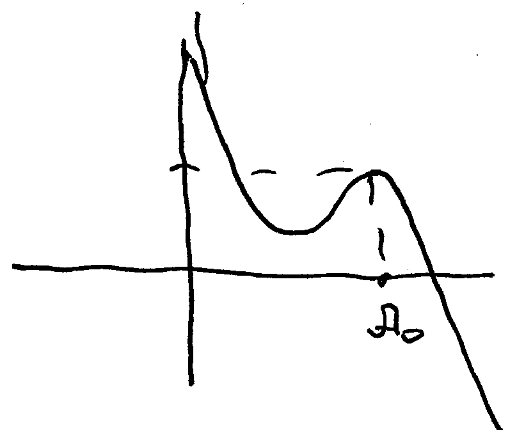
$$A_{xx} + \frac{\partial V}{\partial A} = 0$$

$$V = \left(\tfrac{1}{2}v^2 + A_0^2\right)A^2 - \frac{\lambda}{2}A^4 + \frac{1}{2}\frac{v^2 A_0^4}{A^2}$$



let $v = 0$

For arbitrary v



This is a motion of the particle in the spherically symmetric field

$$\frac{1}{2} A_x^2 + V(A) = E$$

for the solitonic solution

$$E = V(A_0)$$

$$E = \left(\frac{1}{2} V^2 + A_0^2\right) A_0^2 - \frac{1}{2} A_0^4 + \frac{1}{2} V^2 A_0^2 = V^2 A_0^2 + \frac{1}{2} A_0^4$$

We end up with the following first order ODE

$$\frac{1}{2} A_x^2 + \frac{1}{2} V^2 \left(A^2 - \cancel{A_0^2} + \frac{A_0^4}{A^2} - 2A_0^2\right) - \frac{1}{2} \left(A^2 - A_0^2\right)^2 = 0$$

This equation can be exactly integrated, describing " the dark solitons.

In the simplies case $V = 0$

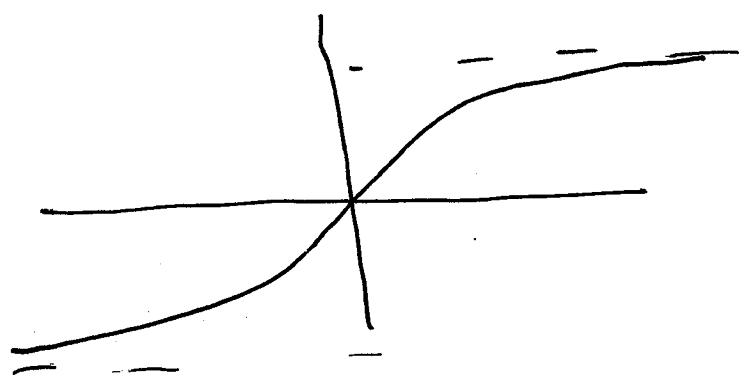
one obtains

$$A_x^2 = (A_0^2 - A^2)^2$$

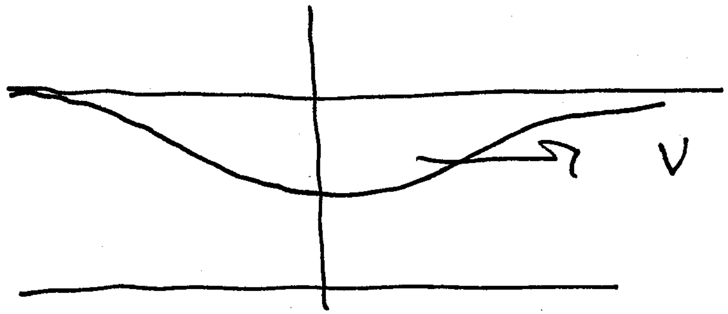
This equation has a solution

$$A = A_0 \tanh A_0 (x - x_0)$$

This is a kink (resting)



In a general case the dark soliton has a form



It could propagate with an arbitrary velocity less than the sound velocity $V < A_0$

When $V \to A_0$, the soliton is described by the KdV equation