Lecture two

Variational derivatives.

The technique of variational derivatives will be used in this course systematically. Let $f(x)$ is a smooth real function defined on the intervale $a < x < b$

Let $H[f]$ is a functional. If this is a linear functional

$$H[f] = \int_a^b g(x) f(x) dx \tag{2.1}$$

This $g(x)$ is an another function. Notice that for smooth f g can be a generalized function, for instanse $g(x) = A\delta(x-\lambda)$

Then

$$H[f] = A f(\lambda)$$

By definition $g(x)$ is a variational derivative of the functional H.

Let us consider a more general, nonlinear functional

$$g(x) = \frac{\delta H}{\delta f} \qquad (2.2)$$

$$H = \int_a^b g(x)\, F(f)\, dx$$

One can add to f a small increment

$$f \to f + \varepsilon\,\delta f \qquad H \to H_\varepsilon$$

Let us calculate $H' = \lim_{\varepsilon \to 0} \frac{H_\varepsilon - H}{\varepsilon}$

Apparently

$$H' = \int_a^b g(x)\, F'(f(x))\, \delta f(x)\, dx$$

H' is a linear functional with respect to variation δf. Hence

$$g(x)\, F'(f(x)) = \frac{\delta H}{\delta f} \qquad (2.3)$$

Thereafter we will assume that $\delta f|_a = \delta f|_b = 0$

let $H = \int_a^b F(x, f, f')\,dx$

adding of increment to f means adding an increment to f'

$f \to f + \varepsilon\delta f \qquad f' \to f' + \varepsilon\delta f'$.

To get rid of $\delta f'$ one must perform integrating by parts

as a result

$$\frac{\delta H}{\delta f} = \frac{\partial F}{\partial f} - \frac{\partial}{\partial x}\frac{\partial F}{\partial f'} \qquad (2.4)$$

In the same way, if

$$H = \int_a^b F(x, f, f', f'')\,dx$$

$$\frac{\delta H}{\delta f} = \frac{\partial F}{\partial f} - \frac{\partial}{\partial x}\frac{\partial F}{\partial f'} + \frac{\partial^2}{\partial \lambda^2}\frac{\partial F}{\partial f''} \qquad (2.5)$$

Suppose that I depends ~~of~~ on two functions $\rho = \rho(x)$ and $\varphi = \varphi(x)$ and

$$I = \frac{1}{2}\int_0^L \rho \left(\frac{\partial \varphi}{\partial x}\right)^2 dx + \int_0^L \mathcal{E}(\rho)\, d\rho \tag{2.6}$$

Then

$$\frac{\delta I}{\delta \rho} = \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + \mathcal{E}'(\rho) \tag{2.7}$$

$$\frac{\delta I}{\delta \varphi} = -\frac{\partial}{\partial x}\rho \frac{\partial \varphi}{\partial x} \tag{2.8}$$

let us consider the Hamiltonian ~~of~~ system

$$\frac{\partial \rho}{\partial t} = \frac{\delta H}{\delta \varphi} \qquad \frac{\partial \varphi}{\partial t} = -\frac{\delta H}{\delta \rho} \tag{2.9}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \rho v = 0 \qquad (*)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \varepsilon''(\rho) \frac{\partial \rho}{\partial x} = 0 \qquad (2.10)$$

The second equation can be rewritten as follows

$$\rho \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) + \frac{\partial P}{\partial x} = 0 \quad (*) \qquad (2.11)$$

$$P = P(\rho) \qquad \frac{\partial P}{\partial \rho} = \rho \varepsilon''(\rho)$$

Equations $(*)$ $(**)$ compose Euler equations for compressible fluid in one-dimensional geometry. ρ – is density v – horisonal velocity P – is pressure. We assume that $P = P(\rho)$. Such fluid is called <u>barotropic</u>.

We start with the Euler equations

$$\rho_t + \frac{\partial}{\partial x}\rho V = 0 \qquad \lambda(\rho) = \varepsilon''_\rho$$

$$V_t + V\frac{\partial V}{\partial x} + \lambda(\rho)\rho_x = 0$$

Let us try two functions $\mathcal{U} = \mathcal{U}(\rho, V)$ and $\mathcal{V} = \mathcal{V}(\rho, V)$ such that in virtue of the Euler equations

$$\frac{\partial \mathcal{U}}{\partial T} + \frac{\partial \mathcal{V}}{\partial x} = 0$$

It means that

$$\frac{\partial \mathcal{U}}{\partial \rho}\rho_t + \frac{\partial \mathcal{U}}{}$$

Three cases are interesting

1. $\gamma = 2$ $\qquad \mathcal{E} = \rho$

2. $\gamma = 3$ $\qquad \mathcal{E} = \frac{\rho}{2}$

This is a „superintegrable case" (see later)

3) $\gamma = -1$ $\qquad p = -\frac{1}{\rho}$ $\qquad \mathcal{E} \simeq \frac{1}{2\rho}$

This is the Chaplygin gas. Energy is positive, the pressure is negative. A good model for „dark matter"

Now we show that the system of equation (+) (*) has infinite number of motion constants (integrals).

Let $\varphi = \varphi(\rho, v)$ $\quad \psi = \psi(\rho, v)$ – two functions on (ρ, v) satisfying ~~the~~, in virtue of (+, *), the conditions

$$\frac{\partial \varphi}{\partial t} + \frac{\partial \psi}{\partial x} = 0$$

$$\frac{\partial \varphi}{\partial \rho}(v\rho_x + \rho v_x) + \frac{\partial \varphi}{\partial v}(v v_x + \varepsilon''(\rho)\cdot\rho_x) = \frac{\partial \psi}{\partial \rho}\rho_x + \frac{\partial \psi}{\partial v} v_x$$

Functions φ, ψ satisfy to the following system of linear PDE

$$\left|\begin{aligned} v\frac{\partial \varphi}{\partial \rho} + \varepsilon''\frac{\partial \varphi}{\partial v} &= \frac{\partial \psi}{\partial \rho} \\ \rho\frac{\partial \varphi}{\partial \rho} + v\frac{\partial \varphi}{\partial v} &= \frac{\partial \psi}{\partial v} \end{aligned}\right.$$

Excluding ψ by cross-differentiation we end up with the following equation imposed on φ

$$\rho\frac{\partial^2 \varphi}{\partial \rho^2} = \varepsilon''(\rho)\frac{\partial^2 \varphi}{\partial v^2} \qquad (**)$$

Apparently $I = \int \varphi\, dx$ is the motion constant

In particular

$\varphi = \rho V$ — density of momentum

$\varphi = \frac{1}{2}\rho V^2 + \varepsilon(\rho)$ — density of energy

$\varphi = \rho$ — density of mass

Less trivial integral

$\varphi = V$

$\varphi = \frac{1}{6} V^3 + A(\rho) V \qquad \rho A''(\rho) = \varepsilon''(\rho)$

Let us show that all integrals commute

Suppose A, B are solutions of equation $(**)$

One can define the Poisson bracket as follows

$$\{A, B\} = -\int \left\{ \frac{\partial A}{\partial \rho} \frac{\partial}{\partial x} \frac{\partial B}{\partial V} - \frac{\partial B}{\partial \rho} \frac{\partial}{\partial x} \frac{\partial A}{\partial V} \right\} dx$$

$$\frac{\partial}{\partial x} \frac{\partial A}{\partial V} = \frac{\partial^2 A}{\partial \rho \partial V} \rho_x + \frac{\partial A}{\partial V^2} V_x$$

$$\frac{\partial}{\partial x} \frac{\partial B}{\partial x} = \frac{\partial^2 B}{\partial \rho \partial V} \rho_x + \frac{\partial^2 B}{\partial V^2} V_x$$

We end up with the system of equations

$$\frac{\partial B}{\partial \rho} \frac{\partial^2 A}{\partial \rho \partial V} - \frac{\partial A}{\partial \rho} \frac{\partial^2 B}{\partial \rho \partial V} = \frac{\partial C}{\partial \rho}$$

$$\frac{\partial B}{\partial \rho} \frac{\partial^2 A}{\partial V^2} - \frac{\partial A}{\partial \rho} \frac{\partial B}{\partial V^2} = \frac{\partial C}{\partial V}$$

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By equialize ~~the~~ cross derivatives one get

$$\frac{\partial}{\partial V}\left[\frac{\partial B}{\partial \rho}\frac{\partial^2 A}{\partial \rho \partial V} - \frac{\partial A}{\partial \rho}\frac{\partial^2 B}{\partial \rho \partial V}\right] = \frac{\partial}{\partial \rho}\left(\frac{\partial B}{\partial \rho}\frac{\partial^2 A}{\partial V^2} - \frac{\partial A}{\partial \rho}\frac{\partial^2 B}{\partial V^2}\right)$$

after opening paranthesises most of terms are cancelled and we finish with a very symple equation

$$\frac{\partial^2 B}{\partial \rho^2}\frac{\partial^2 A}{\partial V^2} - \frac{\partial^2 A}{\partial \rho^2}\frac{\partial^2 B}{\partial V^2} = 0$$

which is satisfied in virtue of $[*, *]$ for any arbitrary $\varepsilon(\rho)$

Thus
$\{A, B\} = 0$ – all integrals commute