~~Lecture 4~~ Lecture 3

Simple waves in Hydrodynamics

Let us consider the system of the Euler equation for the compressible fluid

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) &= 0 \\ \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \lambda(\rho)\frac{\partial \rho}{\partial x} &= 0 \\ \lambda(\rho) = \frac{1}{\rho}\frac{\partial P}{\partial \rho} & \end{aligned} \tag{1.1}$$

We assume that fluid is *barotropic* and presume that it depends only on density $P = P(\rho)$.

Note that

$$\frac{\partial P}{\partial \rho} = c^2(\rho) \qquad c \text{ — sound velocity}$$

Let us study a special class of solutions of the system (1.1) when velocity is defined by density

$$v = v(\rho)$$

Now density satisfies the equation

$$\left(\frac{\partial}{\partial t} + S\frac{\partial}{\partial x}\right)\rho = 0 \tag{1.2}$$

$$S = \frac{\partial}{\partial \rho}(\rho v(\rho))$$

$v(\rho)$ is still unknown. To find it we should study second equation which takes form

$$\frac{\partial v}{\partial \rho}\left(\frac{\partial \rho}{\partial t} + v(\rho)\frac{\partial \rho}{\partial x}\right) + \lambda(\rho)\frac{\partial \rho}{\partial x} = 0 \tag{1.3}$$

Equations (1.2) and (1.3) must coincide. Hence

$$\frac{\partial}{\partial \rho}(v\rho) = v + \frac{\lambda}{\frac{\partial v}{\partial \rho}}$$

or

$$(\frac{\partial v}{\partial \rho})^2 = \frac{1}{\rho}\lambda(\rho) = \frac{c^2}{\rho^2}$$

$$\frac{\partial v}{\partial \rho} = \pm\frac{c}{\rho}$$

$$v = \pm\int_{\rho_0}^{\rho}\frac{c}{\rho}d\rho \qquad \rho_0\text{-some density} \tag{1.4}$$

Now

$$S_{\pm} = v \pm c(\rho) \tag{1.5}$$

For the special case of polytropic gas:

$$P = \frac{1}{\gamma}c_0^2\rho_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$c^2 = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$$

c_0 is the sound velocity if $\rho = \rho_0$

$$c = c_0\left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}}$$

Then

$$S_+ = \frac{\gamma+1}{\gamma-1}c_0\left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}} - \frac{2}{\gamma-1}c_0 \tag{1.6}$$

$$v(\rho) = \frac{2}{\gamma - 1} c_0 \left[\left(\frac{\rho}{\rho_0} \right)^{\frac{\gamma-1}{2}} - 1 \right]$$

Suppose that the density variation is small

$$\rho = \rho_0 + \delta\rho$$

$$S_+ = S_0 + S_1 \delta\rho$$

$$S_0 = c_0 \qquad S_1 = \frac{\gamma+1}{2} \frac{c_0}{\rho_0} \tag{1.7}$$

For small deviations from mean density ρ_0 equation (1.2) reads

$$\frac{\partial}{\partial t}(\delta\rho) + (S_0 + S_1 \delta\rho) \frac{\partial}{\partial x} \delta\rho = 0 \tag{1.8}$$

This is the Hopf equation. Coefficient S_1 changes sign if $\gamma < -1$.

Note, that

$$S_- = -c_0 + \frac{3-\gamma}{2} \frac{c_0}{\rho_0} \delta\rho$$

One can obtain the same results by another way. Let us try to find a function of two variables

$$A = A(\rho, v)$$

obeying the equation

$$\frac{\partial A}{\partial t} + S \frac{\partial A}{\partial x} = 0 \tag{1.9}$$

Equation (1.9) can be rewritten as follows:

$$\frac{\partial A}{\partial \rho} \frac{\partial \rho}{\partial t} + \frac{\partial A}{\partial v} \frac{\partial v}{\partial t} + S \left(\frac{\partial A}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial A}{\partial v} \frac{\partial v}{\partial x} \right) = 0$$

taking time derivative from (1.1) one gets

$$\left(v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} \right) \frac{\partial A}{\partial \rho} + \left(v \frac{\partial \rho}{\partial x} + \lambda \frac{\partial \rho}{\partial x} \right) \frac{\partial A}{\partial v} = S \left(\frac{\partial A}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial A}{\partial v} \frac{\partial v}{\partial x} \right) \tag{1.10}$$

Coefficients before $\frac{\partial \rho}{\partial x}$, $\frac{\partial v}{\partial x}$ must vanish. Hence we obtain

$$
\begin{aligned}
\lambda\frac{\partial A}{\partial v} &= (S-v)\frac{\partial A}{\partial \rho}\\
\rho\frac{\partial A}{\partial \rho} &= (S-v)\frac{\partial A}{\partial v}
\end{aligned}
\tag{1.11}
$$

Compatibility condition for system(1.11) gives $(S-v)^2 = \lambda\rho = c^2$. There are two solutions

$$
\begin{aligned}
S_\pm &= v \pm c\\
A &= v + f(\rho)\\
f(\rho) = \int_{\rho_0}^{\rho}\frac{c}{\rho'}d\rho' \quad A_\pm &= v \pm \int_{\rho_0}^{\rho}\frac{c}{\rho'}d\rho'
\end{aligned}
\tag{1.12}
$$

Thus we have following equations

$$
\frac{\partial A_\pm}{\partial t} + S_\pm\frac{\partial A_\pm}{\partial x} = 0 \tag{1.13}
$$

Equations (1.13) present another form of the initial system (1.1). Functions $A_\pm$ are called Riemann's invariants. Suppose that $A_- = 0$. Then

$$
v = \int_{\rho_0}^{\rho}\frac{c}{\rho'}d\rho'
$$

in accordance with (1.4). This solution is called "simple wave".

Let $p \simeq \rho^3$ $\quad c = \rho$

$A_\pm = v \pm \rho \qquad S_\pm = v \pm \rho$

$S_\pm = A_\pm$

The system splits to two independent systems

$$
\frac{\partial A_\pm}{\partial t} + A_\pm\frac{\partial A_\pm}{\partial x} = 0
$$

This is the superintegrable case.