Lecture 4

In the last lecture we cousilered the considered the polytropic gas in which pressure is the power function of density

$$P = \frac{1}{\gamma} c_0^2 \rho_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$ c_0 – sound velocity for $\rho = \rho_0$

We found that the gas-dynamic equations has a special solution – the "simple wave"

In the simple wave density satisfies the equation

$$\frac{\partial \rho}{\partial t} + S(\rho)\frac{\partial \rho}{\partial x} = 0 \qquad (4.0)$$

where

$$S(\rho) = \frac{\gamma+1}{\gamma-1} c_0 \left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}} - \frac{2}{\gamma-1} c_0$$

$S(\rho_0) = c_0$

Suppose that $\quad u \ll 1$

$$\rho = \rho_0(1 + u)$$

Then

$$S = c_0\left(1 + \frac{\gamma+1}{2}u + \dots\right)$$

Hence dimensionless quantity u satisfies to equation

$$\frac{\partial u}{\partial t} + c_0\left(1 + \frac{\gamma+1}{2}u\right)\frac{\partial u}{\partial x} = 0$$

One can go to the ~~moving~~ frame moving with the sound velocity c_0 and introduce the "slow time" τ

such that

$$\frac{\partial u}{\partial t} + c_0\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \tau}$$

Then

$$\frac{\partial u}{\partial \tau} + c_0 \frac{\gamma+1}{2} u \frac{\partial u}{\partial x} = 0$$

Let us denote

$$w = \frac{\gamma+1}{2} c_0 u$$

w_0 has dimension of velocity!

The new function w satisfies to equation

$$\frac{\partial w}{\partial \tau} + w \frac{\partial w}{\partial x} = 0 \qquad (*)$$

This equation is known as „the Hopf equation)

Equation (5.1) is a subject for initial condition

$$w\Big|_{\tau=0} = F(x)$$

~~This~~ A solution of this initial-value problem can be found in the implicit form as follows

$$w(x,t) = F\left(x - w(x,\tau)\cdot\tau\right) \qquad (4.2)$$

Indeed

$$w_x = F'(1 - \tau w_x) \qquad (\cancel{5.2})$$

$$w_x = \frac{F'}{1+\tau F'}$$

$$w_\tau = F'(-w \cancel{+} i w_\tau)$$

$$w_\tau = -\frac{w F'}{1+\tau F'}$$

$$w_t + w w_x = 0$$

Let $F(x) = \frac{1}{1+x^2}$

Then $w(x,\tau)$ is a solution of the cubic equation

$$w\left(1+(x-w\tau)^2\right) = 1$$

Then

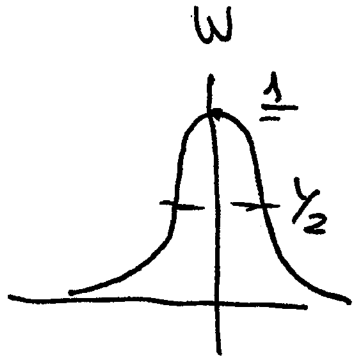
$$F'(x) = -\frac{2x}{(1+x^2)^2} \qquad F''(x) = -2\,\frac{x^2-1}{(x^2+1)^3}$$

at $x = 1$ $\quad F'' = 0 \quad F' = -\frac{1}{2} \quad F = \frac{1}{2}$

F' reaches the minimum

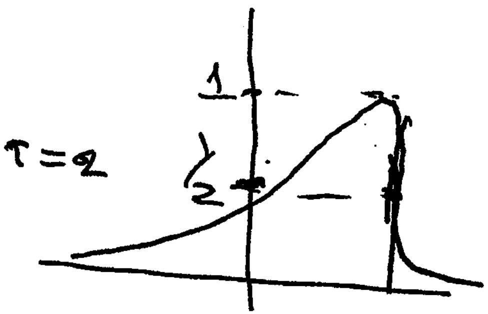
$$w_x\Big|_{w=\frac{1}{2}} = \frac{F'}{1+\tau F'} = \frac{1}{2}\,\frac{1}{1-\frac{\tau}{2}} = \frac{1}{2-\tau}$$

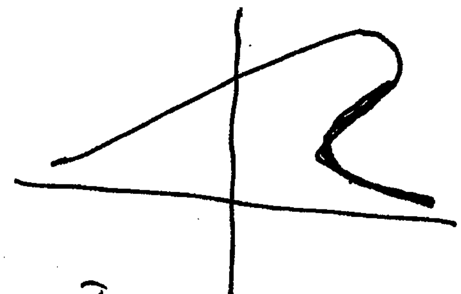
in the moment $\tau = 2$ $\quad w_x = \infty$



$W(x)\big|_{\tau=0}$

$W\big|_{\tau=2}$





$\tau > 2$

This is the "gradient ~~catastrophe~~ catastrophe"

There are two ways to avoid the catastrophe

(1) In the dissipative medium one can replace the Hopf equation by the Burgers equation

$$\frac{\partial W}{\partial \tau} + W\frac{\partial W}{\partial x} = \varepsilon \frac{\partial^2 W}{\partial x^2}$$

$\varepsilon \to 0$

small parameter

One can seek a solution of this equation in a form of propagating wave

$$w = w(x - st)$$

$$-s\frac{\partial w}{\partial x} + \frac{1}{2}\frac{\partial}{\partial x} w^2 = \varepsilon \frac{\partial^2 w}{\partial x^2}$$

We will integrate this equation assuming that $w \to 0$ as $x \to \infty$

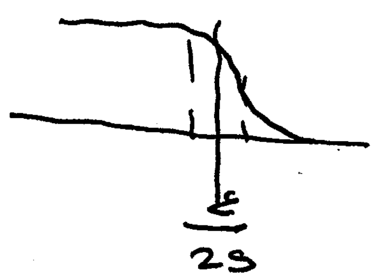
$$-sw + \frac{1}{2} w^2 = \varepsilon \frac{\partial w}{\partial x}$$

at $x \to \infty$ $\frac{\partial w}{\partial x} \to 0$. and

$$w \to 2s$$

The ~~sol~~ equation has a solution

$$w = s\left(1 - \tanh\frac{sx}{2\varepsilon}\right)$$



– shock wave

Thikness of the shock wave is inverse proportional to its intensivity

Solution of hydrodynamic equation in the Riemann invariant.

Let initial equation

$$\rho_t + \frac{\partial}{\partial x}\rho V = 0$$

$$V_t + V V_x + \frac{c^2}{\rho}\rho_x = 0$$

are presented in the form

$$\frac{\partial A^{\pm}}{\partial t} + S^{\pm}(A^+, A^-)\frac{\partial A^{\pm}}{\partial x} = 0 \qquad (+)$$

The solution can be found in the implicit form

Le $A^+ = F\left(x - S^+(A^+, A^-)t\right)$

$$A^- = G\left(x - S^-(A^+, A^-)t\right.$$

$F \neq F_{\xi}$ $\quad G = G(\eta)$ – arbitrary functions of one variable

Then

$$A^+_t = F'(z)\left[- s^+ \cdot - \frac{\partial s^+}{\partial A^+} A^+_t - \frac{\partial s^+}{\partial A^-} A^-_t\right]$$

$$A^-_t = g'\left[- s^- - \frac{\partial s^-}{\partial A^+} A^+_t - \frac{\partial s^-}{\partial A^-} A^-_t\right]$$

The column $\begin{matrix} A^+_t \\ A^-_t \end{matrix}$ is a solution of the

$$\left\| \begin{matrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{matrix} \right\| \begin{bmatrix} A^+_t \\ A^-_t \end{bmatrix} = - \begin{bmatrix} F'(z) s^+ \\ g'(\eta) s^- \end{bmatrix} \qquad (*)$$

$$M_{11} = 1 + F' \frac{\partial s^+}{\partial A^+} \qquad M_{12} = F' \frac{\partial s^+}{\partial A^-}$$

$$M_{21} = g' \frac{\partial s^-}{\partial A^+} \qquad M_{22} = 1 + g' \frac{\partial s^-}{\partial A^-}$$

x- derivatives A^+_x, A^-_x obey almost the same system

$$\left\|\begin{matrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{matrix}\right\| \begin{bmatrix} A^+_x \\ \bar{\Lambda}_x \end{bmatrix} = \begin{bmatrix} F'(\xi) \\ g'(\eta) \end{bmatrix} \qquad (**)$$

Comparing (*) and (**) we see that equations (+) are satisfied.

Hence we found a general solution of the system of Hydrodynamic equation (4.0)