__Lecture 5__ (The Dressing Method)

Let $F = f + ig$ is a complex valued function of two variables u, v. The function F is analytic if the Cauchy–Riemann conditions hold

$$\frac{\partial f}{\partial u} - \frac{\partial g}{\partial v} = 0 \qquad \frac{\partial f}{\partial v} + \frac{\partial g}{\partial u} = 0 \tag{5.1}$$

By introducing complex notation $\lambda = u + iv$ one can rewrite condition (5.1) as follows

$$\frac{\partial F}{\partial \bar\lambda} = 0 \tag{5.2}$$

The function obeying this condition on the whole $\lambda, \bar\lambda$ plane is an entire function. Suppose that F is globally bounded

$$|F| < C \quad \text{for all } \lambda. \tag{5.2}$$

According to the Loiville theorem, this function is identically constant. If $F = P(\lambda)$ – a ~~plyn~~ polynomial $\frac{\partial F}{\partial \bar\lambda} = 0$.

Let us calculate $\frac{\partial}{\partial \bar\lambda} \frac{1}{\lambda}$

$$\frac{\partial}{\partial \bar\lambda} \frac{1}{\lambda} = \frac{\partial}{\partial \bar\lambda} \lim_{\varepsilon \to 0} \frac{\bar\lambda}{|\lambda\bar\lambda| + \varepsilon} = \lim_{\varepsilon \to 0} \frac{\varepsilon}{(|\lambda|^2 + \varepsilon)^2} = \frac{C\delta(u)\delta(v)}{\cancel{C\delta(u)\delta(v)}}$$

$$c = \int \frac{\varepsilon}{(r^2+\varepsilon^2)^2}\, d\varphi \cdot r\, dr \qquad r = \sqrt{u^2+v^2} \qquad y = r^2$$

$$= 2\pi\varepsilon \int_0^\infty \frac{dy}{(y+\varepsilon)^2} = \pi \qquad \cancel{dy = r\,dr}$$

Finally one obtains

$$\frac{\partial}{\partial\bar\lambda}\frac{1}{\lambda} = \pi\,\delta(u)\delta(v) \qquad \frac{\partial}{\partial\bar\lambda}\frac{1}{\lambda-\lambda_0} = \pi\,\delta(u-u_0)\,\delta(v-v_0) \qquad (5.4)$$

This equality is known as Poincare formula

We call a function $F(\lambda,\bar\lambda)$ normalized quasianalytic of

$$\frac{\partial F}{\partial\bar\lambda} = f(\lambda,\bar\lambda) \qquad F \to 1 \text{ as } |\lambda| \to \infty \qquad (5.5)$$

By the use of the ~~Pone~~ Poincare formula one can integrate equation (5.5)

$$F = 1 + \frac{1}{\pi}\int \frac{f(\xi,\bar\xi)}{\lambda-\xi}\, d\xi \wedge d\bar\xi \qquad \text{Here } d\xi \wedge d\bar\xi = du\, dv \qquad (5.6)$$

$$\frac{1}{\lambda-\xi} = \lim_{\varepsilon\to 0} \frac{\bar\lambda-\bar\xi}{|\lambda-\xi|^2+\varepsilon^2}$$

In virtue of (5.6) function $F(\lambda,\bar\lambda)$ has ~~an~~ the asymptotic expansion at $\lambda \to \infty$

$$F = 1 + \frac{F_0}{\lambda} + \frac{F_1}{\lambda^2} + \dots \qquad (5.7)$$

$$F_0 = \frac{1}{\pi}\int f(\xi,\bar\xi)\, d\xi \wedge d\bar\xi \qquad (5.8$$

$$F_1 = \frac{1}{\pi}\int \xi f(\xi,\bar\xi)\, d\xi\, d\bar\xi$$

and so on.

Let $\chi(\lambda,\bar\lambda)$ is a quasianalytic function satisfying to equation (which is called " nonlocal $\bar\partial$-problem)

$$\frac{\partial \chi}{\partial \bar\lambda} = \chi * T = \int \chi(\xi,\bar\xi)\, T(\xi,\bar\xi,\lambda,\bar\lambda)\, d\xi \wedge d\bar\xi \qquad (5.9)$$

Here $T(\xi,\bar\xi,\lambda,\bar\lambda)$ is some kernel which so far is a free functional parameter

let this problem is <u>homegenous</u>, and normalized by condition $\chi \to 0$ as $|\lambda| \to \infty$

~~We can see~~ Let us choose the kernel T such that the homogenous $\bar\partial$ – problem has only ~~trivi~~ trivial solution
In other words ~~$\chi \Rightarrow$~~ ~~conditions~~ asymptotics $\chi \sim O\left(\frac{1}{\lambda}\right)$ at $\lambda \to \infty$ implies

$$\chi \equiv 0 \qquad (5.10)$$

in this case the $\bar\partial$ – problem normalized by condition $\chi \to 1$ as λ has a unique solution, satisfying to the integral equation

$$\chi = 1 + \frac{1}{\pi} \int \frac{\chi(\xi, \bar\xi)\, T(\xi, \bar\xi, \eta, \bar\eta)}{\lambda - \eta}\, d\xi \wedge d\bar\xi\, d\eta \wedge d\bar\eta \qquad (5.11)$$

In this case χ has an expansion ~~like~~ (5.7)

$$\chi \to 1 + \frac{\chi_0}{\lambda} + \frac{\chi_1}{\lambda^2} + \ldots$$

$$\chi_0 = \frac{1}{\pi} \int \chi(\xi, \bar\xi)\, T(\xi, \bar\xi, \eta, \bar\eta)\, d\xi \wedge d\bar\xi\, d\eta \wedge d\bar\eta \qquad (5.12)$$

Let us assume that χ and T depend on three additional parameters x, y, t and introduce three commuting differential operators

$$D_1 \chi = \frac{\partial \chi}{\partial x} + i\lambda \chi \qquad (5.13)$$

$$D_2 \chi = \frac{\partial \chi}{\partial y} + i\lambda^2 \chi \qquad (5.14)$$

$$D_3 \chi = \frac{\partial \chi}{\partial t} + 4i\lambda^3 \chi \qquad (5.15)$$

and demand that the kernel T obeys to equations

$$\begin{aligned} \frac{\partial T}{\partial x} + i\lambda T &= i\eta T \\ \frac{\partial T}{\partial y} + i\lambda^2 T &= i\eta^2 T \\ \frac{\partial T}{\partial t} + 4i\lambda^3 T &= 4i\eta^3 T \end{aligned} \qquad (5.16)$$

Equation (5.16) can be resolved as follows

$$T = R\, e^{i(\varphi(\eta) - \varphi(\lambda))}$$

$$\varphi(\lambda) = \lambda x + \lambda^2 y + 4\lambda^3 t \qquad (5.17)$$

$R = R(\xi, \bar{\xi}, \eta, \bar{\eta})$ is a free functional parameter do not depending on x, y, t. We will call it "Dressing function".

The main point of this construction is following. Application of operator D_i to ~~the~~ $\bar{\partial}$-problem (5.9) does not violate it. In other words

$$\frac{\partial}{\partial \lambda} D_i \chi = D_i \chi * T \qquad (5.18)$$

Notice that operators D_i and $\frac{\partial}{\partial \lambda}$ commute.

Let $P(D_1, D_2, D_3)$ is some ~~polynom~~ differential operator polynomial in D_i and including ~~in~~ multiplication to ~~coefficient~~ coefficient from the left side. The coefficient could be functions on ~~(x,t)~~ x, y, t.

In a general case

$$P(D_1, D_2, D_3)\chi \rightarrow P_0(\lambda) + O\left(\frac{1}{\lambda}\right)$$

where $P_0(\lambda)$ is some polynomial

However one can choose coefficient in P by such way that $P_0(\lambda) \equiv 0$ and $P\chi \rightarrow O\left(\frac{1}{\lambda}\right)$ as $\lambda \rightarrow \infty$

In this case $P\chi$ is a solution of the homogenous $\bar{\partial}$ - problem, hence

$$P\chi \equiv 0 \qquad (5.19)$$

We will call an operator annihilating χ "the P-operator"
We will show that there is infinitely many P-operators and construct the simpliest of them explicitly

Let

$$P_1\chi = -iD_2\chi + D_1^2\chi + u\chi = -i\frac{\partial\chi}{\partial y} + \frac{\partial^2\chi}{\partial x^2} + 2i\lambda\frac{\partial\chi}{\partial x} + u\chi \quad (5.20)$$

Here u is still unknown function

Remember that $1 + \frac{\chi_0}{\lambda} + \frac{\chi_1}{\lambda^2} + \dots$ (5.21)

Plugging (5.21) to (5.20) one see that

$$P_1\chi \to u + 2i\frac{\partial\chi_0}{\partial x} + O\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty$$

~~Hence~~, If we put

$$u = -2i\frac{\partial\chi_0}{\partial x} \quad (5.22)$$

$$P_1\chi \to O\left(\frac{1}{\lambda}\right)$$

Hence P_1 is a P-operator (5.23)

$$P_1\chi \equiv 0 \quad (5.23)$$

Now let

$$P_2\chi = (D_3 + 4D_1^2 + VD_1 + W)\chi \quad (5.24)$$

$$P_2\chi = -12\lambda^2 \frac{\partial \chi}{\partial x} + i\lambda\left(12\frac{\partial^2 \chi}{\partial x^2} + V\chi\right) + \frac{\partial \chi}{\partial t} + 4\frac{\partial^3 \chi}{\partial x^3} + V\frac{\partial \chi}{\partial x} + W\chi$$

Let us send $\lambda \to \infty$. Cancelling of λ- terms (5.25) gives

$$V = -12i\frac{\partial \chi_0}{\partial x} = 6u$$

$$W = 12\frac{\partial \chi_1}{\partial x} + i\left(12\frac{\partial^2 \chi_0}{\partial x^2} + V\chi_0\right) = 12\frac{\partial \chi_1}{\partial x} + \cancel{6\left(\frac{\partial \chi_0}{\partial x} + u\chi_0\right)}$$
$$+ 6\left(-\frac{\partial u}{\partial x} + iu\dot{\chi}_0\right)$$

After making this shoice we establish that

$$P_2\chi = 0 \qquad (5.25)$$

We ~~d~~ will not need a rather complication expression for W. We introduce a new function

$$\varphi = \chi e^{i\varphi(\lambda)} \qquad \chi = \varphi e^{-\varphi(\lambda)}$$

Apparently $D_i\chi = \partial_i \varphi e^{-\varphi(\lambda)}$

Moreover, for any Polynomial differential operator P

$$P(D_1, D_2, D_3)\chi = P\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right)\varphi \cdot e^{-\varphi(\lambda)} \qquad (5.26)$$

In particular equations (5.23), (5.25) take forms

$$i\frac{\partial\varphi}{\partial y} = L\varphi \qquad L = \frac{\partial^2}{\partial x^2} + u$$

$$\frac{\partial\psi}{\partial t} + M\psi = 0 \qquad M = 4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + w \tag{5.27}$$

Equations (5.27) are compatible in virtue of the construction presented above. The compatibility contions read

$$L_t - iM_y = [L, M] \tag{5.28}$$

It is convinient to split

$$w = 3u_x + \mu$$

$$M = M_0 + \mu \tag{5.29}$$

$$M_0 = 4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u_x$$

The commutator $[L, M_0]$ can be easily calculated

$$[L, M_0] = \left(\frac{\partial^2}{\partial x^2} + u\right)\left(\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u_x\right) -$$

$$- \left(\frac{\partial^3}{\partial x^3} + 6u\frac{\partial u}{\partial x} + 3u_x\right)\left(\frac{\partial^2}{\partial x^2} + u\right) \qquad (5.30)$$

Opening of brackets leads a very simple result $[L, M_0]$ is a operator multiplication to the scalar function. All ~~deriv~~ derivatives are cancelled (~~check it~~ check this!) as a result

$$[L, M_0] = -6uu_x - u_{xxx} \qquad (5.31)$$

and

$$[L, M] = [L, M_0] + \left(\frac{\partial^2}{\partial x^2} + u\right)\eta - \eta\left(\frac{\partial^2}{\partial x^2} + u\right)$$

$$[L, M] = -6uu_x - u_{xxx} + \eta_{xx} + 2\eta_x\frac{\partial}{\partial x}$$

Then

$$iM_y = 6iu_y\frac{\partial}{\partial x} + 3iu_{xy} + i\eta_y$$

Collecting all together we find that equations (5.28) are equivalent to the system

$$u_t + 6uu_x + u_{xxx} = \eta_{xx} + 3iu_{xy} + i\eta_y$$

$$- 6iu_y = 2\eta_x$$

After simple cancelling one ends up with the remarkably simple system of equations

$$u_t + 6uu_x + u_{xxx} = i\eta_y \qquad (5.32)$$

$$\eta_x = -3iu_y$$

Which is equivalent to the single equation

$$\frac{\partial}{\partial x}\left(u_t + 6uu_x + u_{xxx}\right) = +3u_{yy} \qquad (5.33)$$

This is the equation KP-1

After replacing $y \to iy$ one obtains the KP-2 equation

$$\frac{\partial}{\partial x}\left(u_t + 6uu_x + u_{xxx}\right) = -3u_{yy} \qquad (5.34)$$