Lecture 6

In the last lecture we derived the KP-eq equations (5.33) and (5.44). Thereafter we will assume ~~equatio~~ mentioned $\frac{\partial u}{\partial y} = 0$, $\frac{\partial \psi}{\partial y} = 0$ and simplify equation to the Korteweg-de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0 \qquad (6.1)$$

As far as $\varphi = \chi e^{i\Phi(x)}$

$\varphi \simeq e^{i\lambda^2 y}\psi(x,t)$, where ψ satisfies to

equations (6.2)

$$L\psi = -\lambda^2 \psi$$

$$\frac{\partial \psi}{\partial t} + M_0 \psi = 0$$

remember that (6.3)

$$L\psi = \frac{\partial^2 \psi}{\partial x^2} + u\psi$$

$$M\psi = 4\frac{\partial^3 \psi}{\partial x^3} + 6u\frac{\partial \psi}{\partial x} + 3u_x \psi$$

~~A traditional way to~~ This is exactly the traditional Lax representation of the KdV equation

The standard way to construction exact solutions of leads through development of the scattering theory for the Schrödinger equation (6.2). We will do this shortly but the use of the dressing method makes possible to find much more straightforward way

The condition $\frac{\partial T}{\partial \bar{y}} = 0$ means that T satisfies to equation

$$(\lambda^2 - \eta^2) T(\lambda, \eta) = 0$$

hence

$$T(\lambda, \bar{\lambda}, \eta, \bar{\eta}) = R(\lambda, \bar{\lambda})\, e^{-2i\dot{\varphi}} \chi(-\lambda, -\bar{\lambda}) \qquad (6.4)$$

or

$$\frac{\partial \chi}{\partial \bar{\lambda}} = \chi(-\lambda, -\bar{\lambda})\, R(\lambda, \bar{\lambda})\, e^{-2i\dot{\varphi}} \qquad \varphi = \lambda x + 4\lambda^3 t$$

Here $R(\lambda, \bar{\lambda})$ — the dressing function is still is a free functional parameter

Remember that

$$\chi \to 1 + \frac{\chi_0}{\lambda} + \dots$$

$$u = -2i \frac{\partial \chi_0}{\partial x} \qquad (6.5)$$

Lei $\lambda_n = i\varkappa_n$ — complex numbers posed on the imaginary axis such that

$$\varkappa_n + \varkappa_m \neq 0$$

We will seek solution of the $\bar{\partial}$ — problem (6.4) in the form

$$\chi = 1 + i \sum_{n=1}^{N} \frac{\chi_m}{\lambda - i\varkappa_m} \qquad (6.5)$$

Then

$$\chi_0 = i \sum_{n=1}^{N} \chi_n \qquad u = 2\frac{\partial \chi_0}{\partial x} = 2\frac{\partial}{\partial x}\sum_{n=1}^{N} \chi_n \qquad (6.6)$$

and

$$\overline{\chi(-\bar{\lambda})} = 1 - i \sum \frac{\bar{\chi}_m}{\lambda + i\varkappa_m}$$

Evaluation of this function to to the point $\lambda = i\varkappa_n$ gives

$$\chi(-i\varkappa_n) = 1 - \sum \frac{\chi_m}{\varkappa_n + \varkappa_m}$$

Calculation of the derivative by $\bar{\lambda}$ gives

$$\frac{\partial f}{\partial \bar{\lambda}} = \pi i \sum \gamma_m \delta(\lambda - i\varkappa_m)\,\delta(\bar{\lambda} + i\varkappa_m) \qquad (6.6)$$

To match (6.5 with (6.4), one must choose

$$R(\lambda, \bar{\lambda}) = \pi \sum c_n \delta(\lambda - i\varkappa_n)\,\delta(\bar{\lambda} + i\varkappa_n) \qquad (6.7)$$

Collecting all together we end up with the system of equations

$$\gamma_n + c_n e^{-2\varphi_n} \sum_{n=1}^{N} \frac{\gamma_m}{\varkappa_n + \varkappa_m} = c_n e^{2\varphi_n} \qquad (6.5)$$

$$\varphi_n = \varkappa_n x + 4\varkappa_n^3 t$$

Now

$$\gamma_n = e^{-\varphi_n}\psi_n$$

$$\psi_n + c_n \sum \frac{\gamma_m e^{-(\varphi_n + \varphi_m)}}{\varkappa_n + \varkappa_m} = c_n e^{\varphi_n} \qquad (6.6)$$

Now

$$u = 2\frac{d}{dx} \sum \psi_n e^{-\varphi_n} \qquad (6.7)$$

System (6.5) and (6.6) have identical determinates which will be called Δ.

Let us denote $\xi_n = \frac{\rho_n}{2x_n} e^{2\varphi_n}$

For $N=2$ (6.8)

$$\Delta = 1 + \xi_1 + \xi_2 + \xi_1\xi_2 \frac{(x_1-x_2)^2}{(x_1+x_2)^2}$$

For $N=3$

$$\Delta = 1 + \xi_1 + \xi_2 + \xi_3 + \xi_1\xi_2 \frac{(x_1-x_2)^2}{(x_1+x_2)^2} + \xi_1\xi_3\left(\frac{x_1-x_3}{x_1+x_3}\right)^2 +$$
$$+ \xi_2\xi_3\left(\frac{x_2-x_3}{x_2+x_3}\right)^2 + \xi_1\xi_2\xi_3\left(\frac{x_1-x_2}{x_1+x_2}\right)^2\left(\frac{x_1-x_3}{x_1+x_3}\right)^2\left(\frac{x_2-x_3}{x_2+x_3}\right)^2$$

This is a hint for construction of the ~~determi~~ determinate in a ~~general~~ general case. It consts of 2^N terms. The first is 1, the last one is

$$\xi_1 \ldots \xi \prod_{i \neq j} \frac{(x_i-x_j)^2}{(x_i+x_j)^2}$$

According to the Kramer's rule the solution of this system

$$\psi_n = \frac{\Delta_n}{\Delta}$$

In the determinate Δ_n n-th column is replaced to $\begin{pmatrix} c_1 e^{\varphi_1} \\ c_N e^{\varphi_N} \end{pmatrix}$

According to (1.7)

$$u = 2\frac{d}{dx}\sum_{n=1}^{N} e^{-\varphi_n}\frac{A_n}{A}$$

However, one can easily realise that

$$\sum e^{-\varphi_n} A_n = \frac{dA}{dx}!$$

Finally we obtain the remarkable result

$$u = 2\frac{d^2}{dx^2}\ln A \qquad (6.9)$$

This formula ~~...~~ conceal the really astonishing property. Lest us replace

$$A \to \tilde{A} = A\, e^{ax+bt+c} \qquad (6.10)$$

, a, b, c – arbitrary constants

then $\frac{d^2}{dx^2}\ln\tilde{A} = \frac{d^2}{dx^2}\ln A$

Transformation (9) does not change a solution of KdV! We will use this property generously