Math 488-588

Lecture 7 . Hirota equation

We start with the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \qquad (1)$$

$$u = 2B_x = 2\frac{\partial^2}{\partial x^2}\ln A$$

$$\frac{\partial}{\partial x}\left(B_t + 6B_x^2 + B_{xxx}\right) = 0$$

$$B_t + 6B_x^2 + B_{xxx} = \mu$$

$$B = \frac{A_x}{A} \qquad B_x = \frac{A_{xx}}{A} - \frac{A_x^2}{A^2}$$

$$B_x^2 = \frac{A_{xx}^2}{A^2} - 2\frac{A_{xx}A_x^2}{A^3} + \frac{A_x^4}{A^4}$$

$$B_{xx} = \frac{A_{xxx}}{A} - \frac{A_xA_{xx}}{A^2} - 2\frac{A_xA_{xx}}{A^2} + \frac{2A_x^3}{A^3}$$

$$B_{xx} = \frac{A_{xxx}}{A} - \frac{3A_xA_{xx}}{A^2} + \frac{2A_x^3}{A^3}$$

$$B_{xxx} = \frac{A_{xxxx}}{A} - \frac{A_x A_{xxx}}{A^2} - \frac{3 A_x A_{xxx}}{A^2} - \frac{3 A_{xx}^2}{A^2} +$$

$$+ \frac{6 A_x^2 A_{xx}}{A^3} + \frac{6 A_x^2 A_{xx}}{A^3} - \frac{6 A_x^4}{A^4}$$

$$B_{xxx} = \frac{A_{xxxx}}{A} - \frac{4 A_x A_{xxx}}{A^2} - \frac{3 A_{xx}^2}{A^2} +$$

$$+ \frac{12 A_x^2 A_{xx}}{A^3} - \frac{6 A_x^4}{A^4}$$

$$B_t = \frac{A_{xt} A - A_t A_x}{A^2}$$

We set

$$A_{xt} A - A_t A_x + A A_{xxxx} - 4 A_x A_{xxx} + 3 A_{xx}^2 = \mu A^2 \qquad (2)$$

This is the Hirota equation

Thereafter we will mostly study the case $\mu = 0$ and assume $A = 1 + C$. We obtain that C satisfies to equation

$$C_{xt} + C_{xxxx} + C_{xt}C - C_t C_x - 4 C_x C_{xxxx} + 3 C_{xx}^2 = 0 \tag{3}$$

Let

$$C = e^{-2\varkappa(x - 4\varkappa^2 t - p)} = \xi \tag{4}$$

Direct substitution to (3) ~~sh~~ shows that function

$$A = 1 + \xi$$

satisfies to equation (3). This is a simple solitonic solution

$$A = 2\frac{\partial^2}{\partial x^2}\ln(1+\xi) = \frac{2\varkappa^2}{\cosh^2\varkappa(x - 4\varkappa^2 t - p)} \tag{5}$$

It is easy to check by direct calculation that function

$$A = 1 + \xi_1 + \xi_2 + \left(\frac{x_1 - x_2}{x_1 + x_2}\right)^2 \xi_1 \xi_2 \qquad x_1 + x_2 \neq 0 \tag{6}$$

also is a solution of the Hirota equation if

$$\xi_1 = e^{-2x_1(x - 4x_1^2 t - p_1)} \qquad \xi_2 = e^{-2x_2(x - 4x_2^2 t - p_2)}$$

Let us study asymptotic of this solution at $t \to \pm\infty$ assuming that

$$x_2 > x_1 > 0$$

To do this we establish first the following simple fact. If some function A generates a ~~sol~~ solution of the KdV equation by the standart formula

$$u = 2 \frac{d^2}{dx^2} \ln A$$

Then ~~the~~ the function $\tilde{A} = A\, e^{Ax + Bt + C}$ where A, B, C – arbitrary constants, generate the same solution. Let us denote

$$C_1 = -2\varkappa_1 (x - 4\varkappa_1^2 t - p_1) \qquad C_2 = -2\varkappa_2 (x - 4\varkappa_2^2 t - p_2) \tag{7}$$

Then

$$\Delta = C_2 - C_1 = 2(\varkappa_1 - \varkappa_2)x + 8(\varkappa_2^3 - \varkappa_1^3)t + \text{const} \tag{8}$$

~~Now suppose that C_1 is fixed~~

As far as $\varkappa_2 > \varkappa_1$

$$\Delta \to -\infty \quad \text{at} \quad t \to -\infty \tag{9}$$

$$\Delta \to \infty \quad \text{at} \quad t \to \infty$$

Now we ~~at~~ ~~at~~ consider solution A ~~at~~ on the characteristic straight line $C_1 = \text{const}$

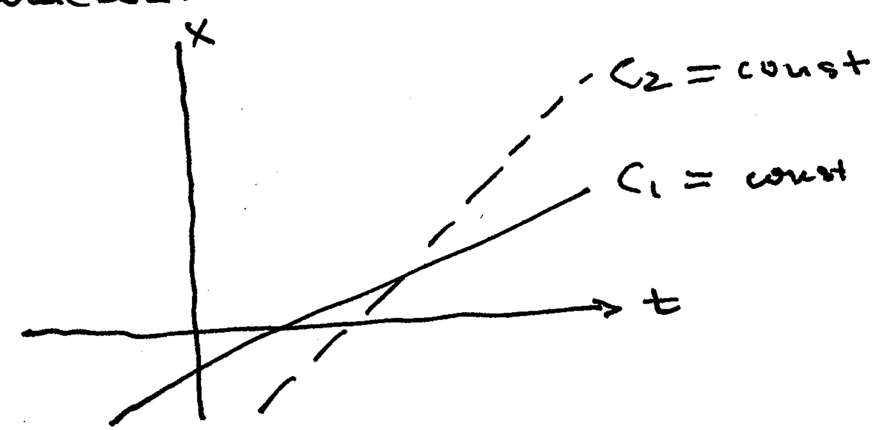

Figure ~~one~~ 1

now as $t \to -\infty$ $\xi_2 \to 0$ and the solitonic solution is

$$A = 1 + \xi_1 \qquad (10)$$

at $t \to -\infty$

$$A \to \xi_2 \left[1 + \frac{(\kappa_1-\kappa_2)^2}{(\kappa_1+\kappa_2)^2} \xi_1 \right] \simeq 1 + \left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2 \xi_1 \qquad (11)$$

ξ_2 is pure exponent and can be cancelled. Comparing (10) and (11) shows the "slow soliton generated by the exponent ξ_1 acquires after interaction with the "fast soliton", generated by the exponent ξ_2 ~~a~~ the negative shift

$$\delta x_1 = \frac{1}{2\kappa_1} \ln \left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2 < 0 \qquad (12)$$

The similar consideration that on the characteristic $c_2 = \text{const}$

$$A \to \left(1 + \frac{(\kappa_1-\kappa_2)^2}{(\kappa_1+\kappa_2)^2} \xi_2\right) \xi_1 \simeq 1 + \frac{(\kappa_1-\kappa_2)^2}{(\kappa_1+\kappa_2)^2} \xi \quad \text{at } t \to -\infty$$

and

$$A \to 1 + \xi_2 \quad \text{at } t \to +\infty$$

It means that the fast soliton acquires the positive shift

$$\delta x_2 = -\frac{1}{2\varkappa_2} \ln\left(\frac{\varkappa_1 - \varkappa_2}{\varkappa_1 + \varkappa_2}\right)^2 > 0 \qquad (13)$$

In fact, the slow soliton turns to the fast soliton. This effect could be ~~to~~ treated as " cammilalism of solitons. As far as shifts of solitons do not depend on their initial positions, ~~the~~ ~~solitons~~ the total soliton shift is an algebraic sum of its shifts due too double ~~colli~~ interactions. If ~~the~~ solitons parameter are ordered as follow

$$0 < \varkappa_1 < \ldots \varkappa_N$$

the total shift of the soliton with index $\varkappa$ is

$$\delta x_k = \frac{1}{2x_k} \sum_{l=1}^{k-1} \ln\left(\frac{x_l + x_k}{x_l - x_k}\right)^2 - \sum_{l=k+1}^{N} \ln\left(\frac{x_l - x_k}{x_l + x_k}\right)^2$$