Math 488-588

Lecture 8 Hirota derivatives

In the lecture 7 we showed that the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \qquad (1)$$

after transformation

$$u = 2\frac{\partial^2}{\partial x^2}\ln A \qquad (2)$$

turns to the Hirota bilinear equation

$$A_{xt}A - A_xA_t + A_{xxxx}A - 4A_xA_{xxx} + 3A_{xx}^2 = \mu A^2 \qquad (3)$$

Now we will present this equation in a more elegant form.

Let $f(x)$, $g(x)$ — is a pair of smooth function. We define the Hirota derivative of n-th order as follows for this pair

$$D_x^n(f,g) = \frac{\partial^n}{\partial y^n} f(x+y)\,g(x-y)\Big|_{y=0} \qquad (3)$$

In particular

$$D_x^1 f\cdot g = f_x g - f g_x$$

$$D_x^2 f\cdot g = f_{xx} g - 2 f_x g_x + f g_{xx} \qquad (4)$$

$$D_x^3 f\cdot g = f_{xxx} g - 3 f_{xx} g_x + 3 f_x g_{xx} - f g_{xxx}$$

$$D_x^4 f\cdot g = f_{xxxx} g - 4 f_{xxx} g_x + 6 f_{xx} g_{xx} - 4 f_x g_{xxx} + f g_{xxxx}$$

Apparently

$$D_x^1 f\cdot f = 0$$

More generally

$$D_x^{2n+1} f\cdot f = 0 \qquad (5)$$

$$D_x^2 ff = 2(f_{xx} f - f_x^2)$$

$$D_x^4 ff = 2(f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2) \qquad (6)$$

In the same way one can define mixed derivatives by two or more variables

$$D_x^m D_t^n f\cdot g = \frac{\partial^m}{\partial y^m}\frac{\partial^n}{\partial \tau^n} f(x+y,\, t+\tau)\, g(x-y,\, t-\tau)\Big|_{\substack{y=0\\ \tau=0}} \qquad (7)$$

In particular

$$D_x^1 D_t^1 f\cdot f = D_{xt}^2 f\cdot f = 2\left(f_{xt} f - f_x f_t\right) \qquad (8)$$

One can see that Hirota equation (3) can be presented as follows

$$\left(D_x D_t + D_x^4\right) A\cdot A = \frac{\mu}{2} A^2 \qquad (9)$$

The central point of the Hirota theory is the theorem about exponents.

Let

$$f = e^{P_1 x + \Omega_1 t + \eta_1} \qquad g = e^{P_2 x + \Omega_2 t + \eta_2}$$

Then

$$D_x^m D_t^n f\cdot g = (P_1 - P_2)^m (\Omega_1 - \Omega_2)^n \cdot f\cdot g \qquad (10)$$

In particular

$$D_x^m D_x^n f\cdot f = 0 \qquad (11)$$

<u>Corollary</u>

Let f, g, h – three arbitrary exponents

Then

$$D_x^m D_t^n \, hf \cdot hg = h^2 D_x^m D_t^n \qquad (12)$$

Even more strong statement holds

Let F, G are two arbitrary functions, while h is exponent

Then

$$D_x^m D_t^n \, hF \cdot hG = h^2 D_x^m D_t^n \, F \cdot G. \qquad (13)$$

In other words, Hirota derivative treat multiplication to an exponent as multiplication to a constant. Using of this fact makes possible

to prove the following important theorem.

Let u is an n – solitonic solution of the KDV equation. ~~Then the det det~~ It can be presented as

$$u = 2\frac{d^2}{dx^2}\ln A, \qquad (14)$$

where A is the determinate of the system (6.5) (Lecture 6). Then A is a solution of the Hirota equation if $\mu = 0$

~~Th~~ For three-solitonic solution it could be checked by a direct calculation. For a general case the proof is more tedious, but later on we will present a really simple proof of this fact.

Let us study now the ~~KE~~ equation KП-2

$$\frac{\partial}{\partial x}\left(u_t + 6uu_x + u_{xxx}\right) + 3u_{yy} = 0 \qquad (15)$$

One can easily check (we omit the details) that u is presented by (14), then A satisfy to 2+1 version of the Hirota equation

$$D_x(D_t + D_{xxx})A\cdot A + 3D_y^2 A\cdot A = 0 \qquad (16)$$

Now suppose that

$$A = 1 + f \qquad (17)$$

where f is an arbitrary solution of two compatible equations

$$f_y = f_{xx} \qquad (18)$$

$$f_t = f_{xxx}$$

Then A is a solution of the Hizrote equation (16)