Math 488-588

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Lecture 9

Knoidal wave

We consider again the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \qquad (1)$$

and will look for a solution ~~p...~~ which presents the propagating wave

$$u = u(x-ct) = u_\xi \qquad \xi = x - ct \qquad (2)$$

now

$$u_t = -cu_\xi \qquad u_x = u_\xi$$

Equation (1) gives

$$\frac{\partial}{\partial \xi}\left(-cu + 3u^2 + u_{xx}\right) = 0 \qquad (3)$$

integration of (3) gives

$$-cu + 3u^2 + u_{xx} = -cp + 3p^2 \qquad (4)$$

Here p is some constant. The transform

$$u = p + v$$

leads to equation

$$v_{xx} + 3v^2 - \tilde{c}\, v = 0 \qquad \tilde{c} = c - 6p$$

One can ~~see that~~ constant p can be removed by
renormalization of velocity c. Thus we can put $p=0$
without loss of generality. The resulting equation

$$u_{xx} + 3u^2 - cu = 0 \tag{5}$$

This equation can be integrated as follows

$$\frac{1}{2} u_x^2 + u^3 - \frac{c}{2} u^2 = E \tag{6}$$

Thereafter we assume $c > 0$

E — is a parameter of integration. If $E = 0$
equation (6) describes ~~the~~ solution. But if $E < 0$
the solution of (6) is a periodic function
This equation can be ~~e~~ easily integrated in terms
of elliptic functions, but we like to solve it by
the use of the Hirota equation. Now it takes
form:

$$H\,AA = \left(-C D_{\xi}^{2} + D_{\xi}^{4}\right) A \cdot A = \mu A^{2} \qquad (7)$$

We will consider that the solution is an even function $u(-x) = u(x)$, hence $A(x) = A(-x)$.

We will look a solution of equation (6) in form of infinite series

$$A = \sum_{n=-\infty}^{\infty} \varphi_n \qquad (8)$$

$$\varphi_n = e^{-4\lambda n^2 + 2k n \xi} \qquad (9)$$

Equation (7) means that

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} L_{nm} = 0 \qquad (10)$$

Here $L_{nm} = H \varphi_n \varphi_m =$

$$= e^{-4\lambda(n^2+m^2) + 2k(n+m)\xi} \left[16k^4 (n-m)^4 - 4ck^2 (n-m)^2 - \mu\right]$$

Now we present

$$2n = p+q \qquad 2m = p-q \tag{11}$$

$$p = n+m \qquad q = n-m \qquad p^2+q^2 = 2(n^2+m^2)$$

p, q are integers. Now $L_{nm} \to L_{pq}$

$$L_{pq} = e^{-2\lambda p^2 + 4kp\xi} \times e^{-2\lambda q^2}\left[16k^4 q^4 - 2ck^2q^2 - \mu\right] \tag{12}$$

Now equation (10) turns to

$$\sum_p \sum_q L_{pq} = 0 \tag{13}$$

But

$$L_{pq} = A(p,\xi)\, B(q) \tag{14}$$

Let us fix p and demand

$$\sum_q B(q) = 0$$

It seems that L_{pq} is factorized, but there is one delicate moment. As far as n, m are integers, p and q must be ~~in~~ integers of the same parity — both even (if n, m are of the same parity) or both odd (if n, m are number ~~of~~ different parities. As a result equation (13) is equivalent to two equations

$$\sum_{q=2l\text{ — all even}} e^{-2\lambda q^2}\left[16k^4q^4 - 2ck^2q^2 - \mu\right] = 0 \qquad (15)$$

$$\sum_{q=2l+1\text{ — all odd}} e^{-2\lambda q^2}\left[16k^4q^4 - 2ck^2q^2 - \mu\right] = 0 \qquad (16)$$

Equations (15) (16) impose two restrictions on three parameters λ, k, μ. One of them, namely λ, can be chosen ~~arbitrary~~ as an arbitrary positive number. If $\lambda \to \infty$, the solution tends to one-solitonic solution. If $\lambda \to 0$ the solution is a weakly nonlinear wave of small amplitude on the constant base.

In ~~&~~ equations (15) one can replace $k \to ik$, $C \to -C$, and they preserve their form

Now we should explain why solution (8) leads to a periodic u. Let us make a shift

$$\xi \to \xi + \frac{4\lambda}{k}$$

Then $\varphi_n \to e^{-4\lambda(n-1)^2 + 2kn\xi} =$

$$= e^{-4\lambda(n-1) + 2k(n-1)\xi} \; e^{2k\xi}$$

As far as n run all integers, $(n-1)$ makes the same thing. As a result $A \to A e^{2k\xi}$, while u is unchanged.

In the same way one can construct two-periodic solutions of the KdV,

This is a good subject for an individual research.