

Solitons in Mathematics and Physics
MATH 488-588

Instructor: V.E. Zakharov

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Lecture 1

Simple waves in Hydrodynamics

Let us consider the system of the Euler equation for the compressible fluid

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) &= 0 \\ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \lambda(\rho) \frac{\partial \rho}{\partial x} &= 0 \\ \lambda(\rho) &= \frac{1}{\rho} \frac{\partial P}{\partial \rho}\end{aligned}\tag{1.1}$$

We assume that fluid is *barotropic* and presume that it depends only on density $P = P(\rho)$.

Note that

$$\frac{\partial P}{\partial \rho} = c^2(\rho) \quad c \text{ — sound velocity}$$

Let us study a special class of solutions of the system (1.1) when velocity is defined by density

$$v = v(\rho)$$

Now density satisfies the equation

$$\left(\frac{\partial}{\partial t} + S \frac{\partial}{\partial x} \right) \rho = 0\tag{1.2}$$

$$S = \frac{\partial}{\partial \rho}(\rho v(\rho))$$

$v(\rho)$ is still unknown. To find it we should study second equation which takes form

$$\frac{\partial v}{\partial \rho} \left(\frac{\partial \rho}{\partial t} + v(\rho) \frac{\partial \rho}{\partial x} \right) + \lambda(\rho) \frac{\partial \rho}{\partial x} = 0 \quad (1.3)$$

Equations (1.2) and (1.3) must coincide. Hence

$$\frac{\partial}{\partial \rho}(v\rho) = v + \frac{\lambda}{\frac{\partial v}{\partial \rho}}$$

or

$$\left(\frac{\partial v}{\partial \rho} \right)^2 = \frac{1}{\rho} \lambda(\rho) = \frac{c^2}{\rho^2}$$

$$\frac{\partial v}{\partial \rho} = \pm \frac{c}{\rho}$$

$$v = \pm \int_{\rho_0}^{\rho} \frac{c}{\rho} d\rho \quad \rho_0 \text{-some density} \quad (1.4)$$

Now

$$S_{\pm} = v \pm c(\rho) \quad (1.5)$$

For the special case of polytropic gas:

$$P = \frac{1}{\gamma} c_0^2 \rho_0 \left(\frac{\rho}{\rho_0} \right)^{\gamma}$$

$$c^2 = c_0^2 \left(\frac{\rho}{\rho_0} \right)^{\gamma-1}$$

c_0 is the sound velocity if $\rho = \rho_0$

$$c = c_0 \left(\frac{\rho}{\rho_0} \right)^{\frac{\gamma-1}{2}}$$

Then

$$S_+ = \frac{\gamma+1}{\gamma-1} c_0 \left(\frac{\rho}{\rho_0} \right)^{\frac{\gamma-1}{2}} - \frac{2}{\gamma-1} c_0 \quad (1.6)$$

$$v(\rho) = \frac{2}{\gamma - 1} c_0 \left[\left(\frac{\rho}{\rho_0} \right)^{\frac{\gamma-1}{2}} - 1 \right]$$

Suppose that the density variation is small

$$\begin{aligned} \rho &= \rho_0 + \delta\rho \\ S_+ &= S_0 + S_1 \delta\rho \\ S_0 &= c_0 \quad S_1 = \frac{\gamma + 1}{2} \frac{c_0}{\rho_0} \end{aligned} \tag{1.7}$$

For small deviations from mean density ρ_0 equation (1.2) reads

$$\frac{\partial}{\partial t}(\delta\rho) + (S_0 + S_1 \delta\rho) \frac{\partial}{\partial x} \delta\rho = 0 \tag{1.8}$$

This is the Hopf equation. Coefficient S_1 changes sign if $\gamma < -1$.

Note, that

$$S_- = -c_0 + \frac{3 - \gamma}{2} \frac{c_0}{\rho_0} \delta\rho$$

One can obtain the same results by another way. Let us try to find a function of two variables

$$A = A(\rho, v)$$

obeying the equation

$$\frac{\partial A}{\partial t} + S \frac{\partial A}{\partial x} = 0 \tag{1.9}$$

Equation (1.9) can be rewritten as follows:

$$\frac{\partial A}{\partial \rho} \frac{\partial \rho}{\partial t} + \frac{\partial A}{\partial v} \frac{\partial v}{\partial t} + S \left(\frac{\partial A}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial A}{\partial v} \frac{\partial v}{\partial x} \right) = 0$$

taking time derivative from (1.1) one gets

$$\left(v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} \right) \frac{\partial A}{\partial \rho} + \left(v \frac{\partial \rho}{\partial x} + \lambda \frac{\partial \rho}{\partial x} \right) \frac{\partial A}{\partial v} = S \left(\frac{\partial A}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial A}{\partial v} \frac{\partial v}{\partial x} \right) \tag{1.10}$$

Coefficients before $\frac{\partial \rho}{\partial x}$, $\frac{\partial v}{\partial x}$ must vanish. Hence we obtain

$$\begin{aligned}\lambda \frac{\partial A}{\partial v} &= (S - v) \frac{\partial A}{\partial \rho} \\ \rho \frac{\partial A}{\partial \rho} &= (S - v) \frac{\partial A}{\partial v}\end{aligned}\tag{1.11}$$

Compatibility condition for system(1.11) gives $(S - v)^2 = \lambda\rho = c^2$. There are two solutions

$$\begin{aligned}S_{\pm} &= v \pm c \\ A &= v + f(\rho) \\ f(\rho) &= \int_{\rho_0}^{\rho} \frac{c}{\rho'} d\rho' A_{\pm} = v \pm \int_{\rho_0}^{\rho} \frac{c}{\rho'} d\rho'\end{aligned}\tag{1.12}$$

Thus we have following equations

$$\frac{\partial A_{\pm}}{\partial t} + S_{\pm} \frac{\partial A_{\pm}}{\partial x} = 0\tag{1.13}$$

Equations (1.13) present another form of the initial system (1.1). Functions A_{\pm} are called Riemann's invariants. Suppose that $A_- = 0$. Then

$$v = \int_{\rho_0}^{\rho} \frac{c}{\rho'} d\rho'$$

in accordance with (1.4). This solution is called "simple wave".

Lecture 2

Dispersion is included

Suppose that the flow is potential

$$v = \nabla\Phi.$$

In any dimensions one can rewrite the continuity equation as follows

$$\frac{\partial\rho}{\partial t} + \text{div}(\rho\nabla\Phi) = 0 \quad (2.1)$$

The Euler equation can be replaced by the Bernulli equation

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 + w(\rho) = 0 \quad (2.2)$$

$$w(\rho) = \int_{\rho_0}^{\rho} \frac{1}{\rho'} \frac{\partial P(\rho')}{\partial \rho'} d\rho'$$

Let us consider energy of the field

$$H = \frac{1}{2} \int \rho(\nabla\Phi)^2 d\vec{r} + \int \varepsilon(\rho) d\vec{r} \quad (2.3)$$

$$\varepsilon(\rho) = \int_{\rho_0}^{\rho} w(\rho') d\rho'$$

and calculate its functional derivatives by ρ , Φ

$$\begin{aligned} \frac{\delta H}{\delta \rho} &= \frac{1}{2}(\nabla\Phi)^2 + w(\rho) \\ \frac{\delta H}{\delta \Phi} &= -\text{div}(\rho\nabla\Phi) \end{aligned} \quad (2.4)$$

One can see that equations (2.2) can be written as follows

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= \frac{\delta H}{\delta \Phi} \\ \frac{\partial \Phi}{\partial t} &= -\frac{\delta H}{\delta \rho}\end{aligned}\quad (2.5)$$

Thus Φ , ρ are canonically conjugated pair of variables. H is Hamiltonian.

Let us us generalize the system (2.1) (2.2) including into consideration dependence of phase velocity on wave number. To do this we add to the Hamiltonian an additional quadratic term

$$\begin{aligned}H &\rightarrow H + H_1 \\ H_1 &= \alpha Lc \int (\nabla \rho \nabla \Phi) d\vec{r} + \frac{\beta}{2} \rho_0 L^2 \int (\Delta \Phi)^2 d\vec{r} + \frac{\gamma L^2 c^2}{2\rho_0} \int (\nabla \rho)^2 d\vec{r}\end{aligned}\quad (2.6)$$

Here L is a characteristic length, α, β, γ are dimensionless constants.

Now equations (2.5) read

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \text{div}(\rho \nabla \Phi) &= -\alpha Lc \Delta \rho + \beta \rho_0 L^2 \Delta^2 \Phi \\ \frac{\partial \Phi}{\partial t} + \frac{1}{2} (\nabla \Phi)^2 + w(\rho) &= \alpha Lc \Delta \Phi + \frac{\gamma L^2 c^2}{\rho_0} \Delta \rho\end{aligned}\quad (2.7)$$

For small perturbations of density one can put

$$w(\rho) \simeq \frac{c_0^2}{\rho_0} \delta \rho$$

(Thereafter we replace $c_0 \rightarrow c$.)

Linearization of equations (2.7) leads to system

$$\begin{aligned}\frac{\partial}{\partial t} \delta \rho + \rho_0 \Delta \Phi &= -\alpha Lc \Delta \delta \rho + \beta \rho_0 L^2 \Delta^2 \Phi \\ \frac{\partial \Phi}{\partial t} + \frac{c^2}{\rho_0} \delta \rho &= \alpha Lc \Delta \Phi + \frac{\gamma L^2 c^2}{\rho_0} \Delta \delta \rho\end{aligned}\quad (2.8)$$

Let us use Fourier harmonics

$$\begin{aligned}\delta\rho(\vec{r}, t) &\rightarrow \delta\rho(\vec{k}, \omega)e^{-i\omega t+i\vec{k}\vec{r}} \\ \Phi(\vec{r}, t) &\rightarrow \Phi(\vec{k}, \omega)e^{-i\omega t+i\vec{k}\vec{r}}\end{aligned}$$

One gets

$$\begin{aligned}-(i\omega + \alpha Lck^2)\delta\rho &= \rho_0 k^2(1 + \beta L^2 k^2)\Phi \\ -(i\omega - \alpha Lck^2)\Phi &= -\frac{c^2}{\rho_0}(1 + \gamma L^2 k^2)\delta\rho\end{aligned}\quad (2.9)$$

Compatibility condition for system (2.9) leads to following dispersion relation:

$$\omega^2 = c^2 k^2[(1 + \beta L^2 k^2)(1 + \gamma L^2 k^2) - \alpha^2 L^2 k^2] \quad (2.10)$$

$$\omega^2 = c^2 k^2[1 + \varepsilon L^2 k^2 + \beta\gamma(L^2 k^2)^2] \quad (2.11)$$

$$\varepsilon = \beta + \gamma - \alpha^2 \quad (2.12)$$

In the limit $(\vec{k}L)^2 \rightarrow 0$ one can put approximately

$$\begin{aligned}\omega^2 &\simeq c^2 k^2(1 + \varepsilon L^2 k^2) \\ \omega &\simeq c|k|(1 + \varepsilon L^2 k^2)^{\frac{1}{2}} \simeq c|k|(1 + \frac{1}{2}\varepsilon L^2 k^2)\end{aligned}\quad (2.13)$$

Suppose now that one direction is preferred

$$\begin{aligned}\vec{k} &= (p, \vec{k}_\perp) \quad |p| \gg |k_\perp| \\ |k| &= \sqrt{p^2 + k_\perp^2} \simeq p + \frac{1}{2}\frac{k_\perp^2}{p}\end{aligned}$$

Finally

$$\omega = c(p + \frac{1}{2}\frac{k_\perp^2}{p} + \frac{1}{2}\varepsilon L^2 p^3) \quad (2.14)$$

To include effects of dispersion in equation for weak simple wave one has to change equation (1.8) by more complicated equation:

$$\frac{\partial}{\partial t}\delta\rho + (c + S_1\delta\rho)\frac{\partial}{\partial x}\delta\rho + \frac{1}{2}c\partial^{-1}\Delta_\perp\delta\rho - \frac{\varepsilon}{2}cL^2\frac{\partial^3}{\partial x^3}\delta\rho = 0 \quad (2.15)$$

Here $\partial^{-1}f(x) = \int f(x)dx$.

Lecture 3

Soliton appears

If $\varepsilon > 0, \omega'' > 0$ this is a case of *positive dispersion*. In the opposite case $\varepsilon < 0$ the dispersion is *negative*. In the most physical situations $S_1 > 0$. Thereafter we will assume that this condition is satisfied. Let us differentiate (2.15) by x and divide by c . One has

$$\frac{\partial}{\partial x} \left\{ \frac{\partial}{c \partial t} \delta \rho + \left(\pm 1 + \frac{S_1}{c} \delta \rho \right) \frac{\partial}{\partial x} \delta \rho \right\} + \frac{1}{2} \Delta_{\perp} \delta \rho - \frac{\varepsilon}{2} L^2 \frac{\partial^4}{\partial x^4} \delta \rho = 0 \quad (3.1)$$

We generalized equation (2.15) a bit, assuming that in the advective term $c \frac{\partial}{\partial x} \delta \rho$ velocity c could have negative sign (see (1.7)).

Now we introduce dimensionless variable:

$$\frac{S_1}{c} \delta \rho = 6u$$

and rescale time and spatial coordinates

$$\begin{aligned} x &\rightarrow L \sqrt{\frac{|\varepsilon|}{2}} x \\ r_{\perp} &\rightarrow L \sqrt{|\varepsilon|} r_{\perp} \\ t &\rightarrow \frac{1}{c} L \sqrt{\frac{|\varepsilon|}{2}} t \end{aligned}$$

We set following dimensionless equation

$$\frac{\partial}{\partial x} \left\{ \frac{\partial u}{\partial t} + (\pm 1 + 6u) \frac{\partial u}{\partial x} \right\} + \Delta_{\perp} u \mp \frac{\partial^4}{\partial x^4} u = 0 \quad (3.2)$$

If there is only one perpendicular coordinate y , one obtains

$$\frac{\partial}{\partial x} \left\{ \frac{\partial u}{\partial t} + (\pm 1 + 6u) \frac{\partial u}{\partial x} \right\} + \frac{\partial^2 u}{\partial y^2} \mp \frac{\partial^4}{\partial x^4} u = 0 \quad (3.3)$$

If we study essentially nonstationary solutions, one can go to the moving frame

$$x \rightarrow x \mp ct$$

Then we get two equations. If $\varepsilon > 0$

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} - \frac{\partial^3}{\partial x^3} u \right) + \frac{\partial^2 u}{\partial y^2} = 0 \quad (3.4)$$

This is KP-1 equation.

If $\varepsilon < 0$ one has

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3}{\partial x^3} u \right) + \frac{\partial^2 u}{\partial y^2} = 0 \quad (3.5)$$

This is KP-2 equation.

In absence of dependence on y KP equations reduce to the Korteweg de Vries equations

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} = 0 \quad (3.6)$$

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \quad (3.7)$$

In fact equations (3.6) and (3.7) are equivalent. Equation (3.7) goes to (3.6) after a simple transform $u \rightarrow -u$, $t \rightarrow -t$.

Stationary solutions of equation (3.3) obey one of Boussinesq equations

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} + 3 \frac{\partial^2}{\partial x^2} u^2 = 0 \quad (3.8)$$

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} + 3 \frac{\partial^2}{\partial x^2} u^2 = 0 \quad (3.9)$$

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} + 3 \frac{\partial^2}{\partial x^2} u^2 = 0 \quad (3.10)$$

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} + 3 \frac{\partial^2}{\partial x^2} u^2 = 0 \quad (3.11)$$

For small distortion δu of solution u_0 one can put

$$u \simeq u_0 + \delta u e^{iqy+ipx}$$

We have following four dispersion relations for distortion:

$$q^2 = p^2 + p^4 \quad (3.12)$$

$$q^2 = p^2 - p^4 \quad (3.13)$$

$$q^2 = -p^2 + p^4 \quad (3.14)$$

$$q^2 = -p^2 - p^4 \quad (3.15)$$

One can see that the trivial solution $u = 0$ is stable only in framework of equation (3.8). In (3.9) instability takes place at $p \rightarrow \infty$. In (3.10) at $p \rightarrow 0$. Equation (3.11) is pure elliptic.

All KP-1, KP-2, KdV and Boussinesq equations, except (3.10) have solitonic solutions. We will study first equation (3.7). We will look for solutions in form of propagating waves

$$u = u(x - Vt) \quad \text{at } t = 0$$

$$-V \frac{\partial u}{\partial x} + 3 \frac{\partial}{\partial x} u^2 + \frac{\partial^3 u}{\partial x^3} = 0$$

u satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + 3u^2 - Vu = \text{const} \quad (3.16)$$

But this constant should be zero in order to give us decaying solution.

This equation can be integrated once

$$\frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 + u^3 - \frac{1}{2} V u^2 = E \quad (3.17)$$

Equation (3.17) can be solved in elliptic functions. We will study it latter on. So far we are interested only in solutions decaying at infinity: $u \rightarrow 0$ at $x \rightarrow \pm\infty$. This condition implies $E = 0$, $V > 0$.

Let $V = 4k^2$. One can check that equation

$$\frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 + u^3 - 2k^2 u = 0 \quad (3.18)$$

has following general solution

$$u = \frac{2k^2}{\cosh^2 k(x - x_0)} \quad (3.19)$$

x_0 -constant of integration.

Finally we obtain

$$u = \frac{2k^2}{\cosh^2 k(x - x_0 - 4k^2 t)} \quad (3.20)$$

This is soliton in a media with negative dispersion. The soliton is a bump, propagating faster then sound, in the right direction in the frame moving with sound velocity.

Equation (3.6) has following solitonic solution

$$u = -\frac{2k^2}{\cosh^2 k(x - x_0 + 4k^2 t)} \quad (3.21)$$

In a medium with positive dispersion soliton is a dip, propagating slower than sound, in the left direction moving with sound velocity.

The Boussinesq equations (3.8)-(3.11) have solitonic solutions in a form

$$u = u(x - \lambda y)$$

At $y = 0$ $u(x)$ satisfies one of four ODE

$$(\lambda^2 - 1)u + \frac{\partial^2 u}{\partial x^2} + 3u^2 = 0 \quad (3.22)$$

$$(\lambda^2 - 1)u - \frac{\partial^2 u}{\partial x^2} + 3u^2 = 0 \quad (3.23)$$

$$(\lambda^2 + 1)u + \frac{\partial^2 u}{\partial x^2} + 3u^2 = 0 \quad (3.24)$$

$$(\lambda^2 + 1)u - \frac{\partial^2 u}{\partial x^2} + 3u^2 = 0 \quad (3.25)$$

Equation (3.22) at $\lambda^2 < 1$ has bump-type solitonic solutions. Equation (3.23) at $\lambda^2 > 1$ has dip-type solitonic solutions. Equation (3.24) has no solitonic solutions. Equation (3.25) has dip-type solitonic solution at any λ .

Now we study solitonic solutions for the KP equations. We will seek them in a form

$$u = u(x - \lambda y - vt)$$

For KP-1 one gets

$$-\frac{\partial^2 u}{\partial x^2} + 3u^2 + (\lambda^2 - v)u = 0$$

The dip-type solitons exist for any λ and any v , satisfying the condition

$$\lambda^2 - v > 0$$

For KP-2 one obtains

$$\frac{\partial^2 u}{\partial x^2} + 3u^2 + (\lambda^2 - v)u = 0$$

Bump-type solitons exist for any λ and any v , satisfying the condition

$$\lambda^2 - v < 0$$

Lecture 4

Solitons for shallow water

Lets us consider a layer of ideal fluid of thickness h . The bottom is solid, the surface is free. Gravitational acceleration g is perpendicular to the bottom, surface tension with coefficient σ is included. The fluid is incompressible, density is equal to unity.

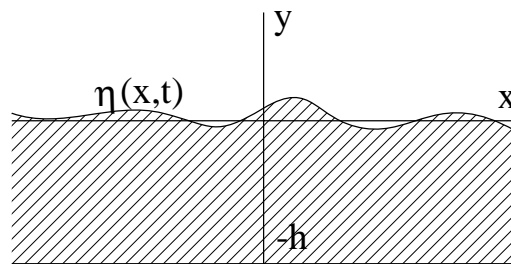


Figure 4.1: Fluid domain.

Surface elevation is $\eta = \eta(x, t)$. Flow of fluid is potential

$$v = \nabla\Phi \quad \text{and} \quad \nabla \cdot v = 0$$

due to incompressibility, potential satisfies the Laplace equation

$$\Delta\Phi = 0 \tag{4.1}$$

Let us denote

$$\Phi|_{y=\eta} = \Psi(x, t) \quad (4.2)$$

apparently

$$\left. \frac{\partial \Phi}{\partial y} \right|_{y=-h} = 0 \quad (4.3)$$

Boundary conditions (4.2), (4.3) define uniquely a solution of the Laplace equation. Thus it is enough to follow the evolution of $\eta(x, t)$, $\Psi(x, t)$

We will not prove following theorem.(The proof is put in application.)

Theorem 1

η, Ψ is a pair of canonical variables They obey equations

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= \frac{\delta H}{\delta \Psi} \\ \frac{\partial \Psi}{\partial t} &= -\frac{\delta H}{\delta \eta} \end{aligned} \quad (4.4)$$

here H – total energy of fluid, consisting of kinetic and potential energy

$$H = T + U \quad (4.5)$$

Potential energy can be found in explicit form

$$U = \frac{1}{2}g \int_{-\infty}^{\infty} \eta^2 dx + \sigma \int (\sqrt{1 + \eta_x^2} - 1) dx \quad (4.6)$$

Kinetic energy is given by formula

$$T = +\frac{1}{2} \int_{-h}^{\eta} \int_{-\infty}^{+\infty} (\nabla \Phi)^2 dy dx = -\frac{1}{2} \int \Psi \cdot \Psi_n ds \quad (4.7)$$

Here Ψ_n is normal derivative of potential.

$$\Psi_n = \int G(s, s') \Psi(s') ds' \quad (4.8)$$

Here $G(s, s') = G(s', s)$ – Green's function for the Dirichlet-Neumann boundary problem. It cannot be expressed in an explicit form for arbitrary $\eta(x, t)$.

However, the Laplace equation can be solved approximately if $k\eta \rightarrow 0$, k – characteristic wave number.

$$\begin{aligned}
\Delta\Phi &= \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \\
\Phi &= \Phi_0 + \Phi_1 + \dots \\
\frac{\partial^2\Phi_0}{\partial y^2} &= 0 \\
\Phi_0 &= \Psi(x, t)
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
\frac{\partial^2\Phi_1}{\partial y^2} &= -\frac{\partial^2\Psi}{\partial x^2} \\
\Phi_1|_{y=\eta} = 0 \quad \frac{\partial\Phi_1}{\partial y}\Big|_{y=-h} &= 0 \\
\Phi_1 &= -\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} \cdot y^2 + C_1y + C_2 \\
-\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2}\eta^2 + C_1\eta + C_2 &= 0 \\
C_1 &= -h\frac{\partial^2\Psi}{\partial x^2} \\
C_2 &= \frac{\partial^2\Psi}{\partial x^2} \left(-\frac{1}{2}\eta^2 - h\eta \right)
\end{aligned} \tag{4.10}$$

Assuming that steepness of the surface is small $k\eta \rightarrow 0$, one can put $C_2 = 0$ because we need only derivatives of Φ .

Thus

$$\begin{aligned}
\Phi_1 &= -\frac{\partial^2\Psi}{\partial x^2} \left(\frac{1}{2}y^2 + hy \right) \\
\frac{\partial\Phi}{\partial x} &\simeq \frac{\partial\Psi}{\partial x} - \left(\frac{1}{2}y^2 + hy \right) \frac{\partial^3\Psi}{\partial x^3} \\
\frac{\partial\Phi}{\partial y} &\simeq -\frac{\partial^2\Psi}{\partial x^2} (h + y) \\
\left(\frac{\partial\Phi}{\partial x} \right)^2 &\simeq \left(\frac{\partial\Psi}{\partial x} \right)^2 - 2\frac{\partial\Psi}{\partial x} \frac{\partial^3\Psi}{\partial x^3} \left(\frac{1}{2}y^2 + hy \right) \simeq \\
&\simeq \left(\frac{\partial\Psi}{\partial x} \right)^2 + 2\left(\frac{\partial^2\Psi}{\partial x^2} \right)^2 \left(+\frac{1}{2}y^2 + hy \right)
\end{aligned}$$

$$\begin{aligned}
& \int_{-h}^{\eta} [y^2 + 2hy + (h + y)^2] dy = \\
& = \int_{-h}^{\eta} \{h^2 + 4hy + 2y^2\} dy = \\
& = h^2(\eta + h) + 2hy^2|_{-h}^{\eta} + \frac{2}{3}y^3|_{-h}^{\eta} = \\
& = h^2\eta + 2h\eta^2 + \frac{2}{3}\eta^3 + h^3 - 2h^3 + \frac{2}{3}h^3 = \\
& = h^2\eta + 2h\eta^2 + \frac{2}{3}\eta^3 - \frac{1}{3}h^3.
\end{aligned}$$

Finally

$$\begin{aligned}
T &= \frac{1}{2} \int \left(\frac{\partial \Psi}{\partial x} \right)^2 (\eta + h) dx + \\
&+ \int \left[h^2\eta \left(1 + 2\frac{\eta}{h} + \frac{2}{3}\frac{\eta^2}{h^2} \right) - \frac{1}{6}h^3 \right] \left(\frac{\partial^2 \Psi}{\partial x^2} \right)^2 dx \quad (4.11) \\
U &\simeq \frac{1}{2}g \int \eta^2 dx + \frac{\sigma}{2} \int \left(\frac{\partial \eta}{\partial x} \right)^2 dx
\end{aligned}$$

Using weakly nonlinear approximation $\eta \ll h$ and $k\eta \rightarrow 0$ one can get

$$\begin{aligned}
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(\eta + h) \frac{\partial^2 \Psi}{\partial x^2} &= \frac{1}{3}h^3 \frac{\partial^4 \Psi}{\partial x^4} \\
\frac{\partial \Psi}{\partial t} + \frac{1}{2} \left(\frac{\partial \Psi}{\partial x} \right)^2 + g\eta &= -\sigma \frac{\partial^2 \eta}{\partial x^2}
\end{aligned} \quad (4.12)$$

This is a system of type (2.1),(2.2)

In the linear approximation

$$\begin{aligned}
-i\omega\eta - hk^2 \left(1 - \frac{1}{3}h^2k^2 \right) \Psi &= 0 \\
-i\omega\Psi + (g + \sigma k^2) \eta &= 0 \\
\omega^2 = ghk^2 \left(1 - \frac{1}{3}h^2k^2 \right) \left(1 + \frac{\sigma}{g}k^2 \right)
\end{aligned}$$

$$\frac{\sigma}{g} > \frac{1}{3}h^2 \quad - \text{positive dispersion}$$

$$\frac{\sigma}{g} < \frac{1}{3}h^2 \quad - \text{negative dispersion}$$

We will show that system (4.12) at $\sigma = 0$ is integrable.

Lecture 5

Lax representation

Let us consider the following overdetermined system of differential equations

$$\frac{\partial \Psi}{\partial y} + \hat{\mathbf{L}}\Psi = 0 \quad (5.1)$$

$$\frac{\partial \Psi}{\partial t} + \hat{\mathbf{A}}\Psi = 0 \quad (5.2)$$

$$\begin{aligned} \hat{\mathbf{L}}\Psi &= \frac{\partial^2 \Psi}{\partial x^2} + U\Psi \\ \hat{\mathbf{A}}\Psi &= 4\frac{\partial^3 \Psi}{\partial x^3} + V\frac{\partial \Psi}{\partial x} + W\Psi \end{aligned} \quad (5.3)$$

Comparing cross-derivatives $\frac{\partial^2 \Psi}{\partial t \partial y} = \frac{\partial^2 \Psi}{\partial y \partial t}$ one obtains

$$\left(\frac{\partial \hat{\mathbf{L}}}{\partial t} - \frac{\partial \hat{\mathbf{A}}}{\partial y} - [\hat{\mathbf{L}}, \hat{\mathbf{A}}] \right) \Psi = 0 \quad (5.4)$$

System (5.1),(5.2) must be compatible – what does that mean? We demand that the common Cauchy problem for equations (5.1),(5.2)

$$\Psi|_{t=0, y=0} = \Psi_0(x) \quad -\infty < x < \infty$$

has at least local solution in some domain on the (y, t) plane for any arbitrary (smooth enough) function Ψ_0 of one variable x .

Now we observe that operator $\hat{\mathbf{T}} = \frac{\partial \hat{\mathbf{L}}}{\partial t} - \frac{\partial \hat{\mathbf{A}}}{\partial y} - [\hat{\mathbf{L}}, \hat{\mathbf{A}}]$ is an ordinary differential operator, annihilating any arbitrary function of one variable. Hence we can cancel Ψ in (5.4) and require:

$$\frac{\partial \hat{\mathbf{L}}}{\partial t} - \frac{\partial \hat{\mathbf{A}}}{\partial y} - [\hat{\mathbf{L}}, \hat{\mathbf{A}}] = 0 \quad (5.5)$$

Commutator $[\hat{\mathbf{L}}, \hat{\mathbf{A}}]$ is a second-order differential operator identically equal to zero. Hence, one can “cancel” Ψ in (5.4)

$$\frac{\partial \hat{\mathbf{L}}}{\partial t} - \frac{\partial \hat{\mathbf{A}}}{\partial y} = [\hat{\mathbf{L}}, \hat{\mathbf{A}}] \quad (5.6)$$

$$[\hat{\mathbf{L}}, \hat{\mathbf{A}}] = \left(2 \frac{\partial V}{\partial x} - 12 \frac{\partial U}{\partial x}\right) \frac{\partial^2}{\partial x^2} + \left(\frac{\partial^2 V}{\partial x^2} - 12 \frac{\partial^2 U}{\partial x^2} + 2 \frac{\partial W}{\partial x}\right) \frac{\partial}{\partial x} + \frac{\partial^2 W}{\partial x^2} - 4 \frac{\partial^3 U}{\partial x^3} - V \frac{\partial U}{\partial x} \quad (5.7)$$

Condition (5.5) implies

$$\begin{aligned} 2 \frac{\partial V}{\partial x} - 12 \frac{\partial U}{\partial x} &= 0 \\ V &= 6U + C \end{aligned} \quad (5.8)$$

Then

$$\begin{aligned} -\frac{\partial V}{\partial y} &= \frac{\partial^2 V}{\partial x^2} - 12 \frac{\partial^2 U}{\partial x^2} + 2 \frac{\partial W}{\partial x} \\ \frac{\partial U}{\partial t} - \frac{\partial W}{\partial y} &= \frac{\partial^2 W}{\partial x^2} - 4 \frac{\partial^3 U}{\partial x^3} - V \frac{\partial U}{\partial x} \end{aligned} \quad (5.9)$$

Let us put

$$W = 3 \frac{\partial V}{\partial x} + q.$$

System (5.9) reads now

$$\begin{aligned} 3 \frac{\partial U}{\partial y} &= -\frac{\partial q}{\partial x} \\ \frac{\partial U}{\partial t} + C \frac{\partial U}{\partial x} + \frac{\partial^3 U}{\partial x^3} + 6U \frac{\partial U}{\partial x} &= \frac{\partial q}{\partial y} \end{aligned} \quad (5.10)$$

It is equivalent to the KP-2 equation

$$\frac{\partial}{\partial x} \left(\frac{\partial U}{\partial t} + C \frac{\partial U}{\partial x} + \frac{\partial^3 U}{\partial x^3} + 6U \frac{\partial U}{\partial x} \right) = -3 \frac{\partial^2 U}{\partial y^2} \quad (5.11)$$

(the only difference is the coefficient 3 which can be excluded by the trivial rescaling of y)

We found that equation (5.11) is a compatibility condition for following pair of a linear PDE

$$\begin{aligned} \pm \frac{\partial \Psi}{\partial y} + \frac{\partial^2 \Psi}{\partial x^2} + U \Psi &= 0 \\ \frac{\partial \Psi}{\partial t} + 4 \frac{\partial^3 \Psi}{\partial x^3} + C \frac{\partial \Psi}{\partial x} + 6U \frac{\partial \Psi}{\partial x} + (3 \frac{\partial U}{\partial x} + q) \Psi &= 0 \end{aligned} \quad (5.12)$$

This overdetermined system is the ‘‘LAX Pair’’ for KP-2 equation. Replacing $\frac{\partial}{\partial y} \rightarrow -\frac{\partial}{\partial y}$ does not change anything. Let us consider following Lax Pair

$$\begin{aligned} \pm i \frac{\partial \Psi}{\partial y} + \frac{\partial^2 \Psi}{\partial x^2} + U \Psi &= 0 \\ \frac{\partial \Psi}{\partial t} + 4 \frac{\partial^3 \Psi}{\partial x^3} + C \frac{\partial \Psi}{\partial x} + 6U \frac{\partial \Psi}{\partial x} + (3 \frac{\partial U}{\partial x} + iq) \Psi &= 0 \end{aligned} \quad (5.13)$$

Here U, q are real functions. A compatibility condition for (5.13) is equation

$$\frac{\partial}{\partial x} \left(\frac{\partial U}{\partial t} + C \frac{\partial U}{\partial x} + \frac{\partial^3 U}{\partial x^3} + 6U \frac{\partial U}{\partial x} \right) = 3 \frac{\partial^2 U}{\partial y^2} \quad (5.14)$$

which can be transformed to KP-1 by already mentioned transform $t \rightarrow -t$, $U \rightarrow -U$. Note that both equations (5.11), (5.14) have bump-type solitons.

If there is no dependence on y , one can put $\Psi \sim e^{-\lambda y}$ and system (5.1), (5.2) goes to following:

$$\begin{aligned} \hat{\mathbf{L}} \phi &= \lambda \phi \\ \frac{\partial \phi}{\partial t} + \hat{\mathbf{A}} \phi &= 0 \end{aligned} \quad (5.15)$$

Now $q = 0$ and

$$\hat{\mathbf{A}} \Psi = 4 \frac{\partial^3 \Psi}{\partial x^3} + 6U \frac{\partial \Psi}{\partial x} + 3 \frac{\partial U}{\partial x} \Psi$$

(we put $C = 0$.) This is the “classical” Lax Pair for KdV equation, which now can be written as follows:

$$\frac{\partial \hat{\mathbf{L}}}{\partial t} = [\hat{\mathbf{L}}, \hat{\mathbf{A}}] \quad (5.16)$$

For the Boussinesq equation one has the system

$$\begin{aligned} \hat{\mathbf{A}}\Psi &= \lambda\Psi \\ \mu \frac{\partial \Psi}{\partial y} &= \hat{\mathbf{L}}\Psi \\ \mu^2 &= \pm 1 \\ \hat{\mathbf{A}}\Psi &= 4\frac{\partial^3 \Psi}{\partial x^3} + 6U\frac{\partial \Psi}{\partial x} + 3\frac{\partial U}{\partial x}\Psi + \mu q\Psi + C\frac{\partial \Psi}{\partial x} \end{aligned} \quad (5.17)$$

Here $C = \pm 1$

Additional remarks

1) One can assume that U is complex and study the complex KdV

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + a\frac{\partial^3 U}{\partial x^3} = 0 \quad (5.18)$$

where a – any complex number.

Suppose that $U = \frac{\partial \Phi}{\partial x} + i\rho$, $a = i\alpha$. Now equation (5.18) is equivalent to the system

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho\frac{\partial \Phi}{\partial x} &= -\alpha\frac{\partial^3 \Phi}{\partial x^3} \\ \frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \Phi}{\partial x}\right)^2 - \frac{1}{2}\rho^2 &= \alpha\frac{\partial^2 \rho}{\partial x^2} \end{aligned} \quad (5.19)$$

This is a badly unstable system with Hamiltonian

$$H = \frac{1}{2}\int \rho\left(\frac{\partial \Phi}{\partial x}\right)^2 dx - \frac{1}{6}\int \rho^3 dx - \frac{\alpha}{2}\int \left(\frac{\partial^2 \Phi}{\partial x^2}\right)^2 + \frac{\alpha}{2}\int \left(\frac{\partial \rho}{\partial x}\right)^2 \quad (5.20)$$

2) One can assume that U, V, W are matrix functions. In this case one gets matrix generalizations of KdV, KP and Boussinesq equations. The most simple is the matrix KdV equation

$$\frac{\partial U}{\partial t} + 3\frac{\partial}{\partial x}U^2 + \frac{\partial^3 U}{\partial x^3} = 0 \quad (5.21)$$

Here U — any $N \times N$ matrix. One can assume that U is symmetric: $U^T = U$

Homework

Construct matrix generalisation of the KP-equation.

Lecture 6

From KdV to NLSE

Let us study again overdetermined compatible pair of equations

$$\frac{\partial \chi}{\partial y} + \hat{\mathbf{L}}\chi = 0 \quad (6.1)$$

$$\frac{\partial \chi}{\partial t} + \hat{\mathbf{A}}\chi = 0 \quad (6.2)$$

where again

$$\hat{\mathbf{L}} = \frac{\partial^2}{\partial x^2} + u$$
$$\hat{\mathbf{A}} = 4\frac{\partial^3}{\partial x^3} + (6u + c)\frac{\partial}{\partial x} + 3\frac{\partial u}{\partial x} + q$$

But now χ is vector in \mathbb{C}^n $\chi = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ while u, q are $n \times n$ matrix functions

on x, t, y , and $c = c(t)$ is a matrix function on t only. One can easily check that equation

$$\frac{\partial \hat{\mathbf{L}}}{\partial t} - \frac{\partial \hat{\mathbf{A}}}{\partial y} = [\hat{\mathbf{L}}, \hat{\mathbf{A}}] \quad (6.3)$$

takes the following form

$$\frac{\partial q}{\partial x} = -3\frac{\partial u}{\partial y} + \frac{1}{2}[c, u] \quad (6.4)$$

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left(c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}c\right) + 3\frac{\partial}{\partial x}u^2 + \frac{\partial^3 u}{\partial x^3} = \frac{\partial q}{\partial y} + [u, q] \quad (6.5)$$

This system is a matrix version of the KP equation. We study first one-dimensional case $\partial u/\partial y = 0$, $\partial q/\partial y = 0$ and assume that c is a scalar. Now equation (6.5) reads

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} + 3 \frac{\partial}{\partial x} u^2 + \frac{\partial^3 u}{\partial x^3} + [q, u] = 0 \quad (6.6)$$

$q = q(t)$ - arbitrary matrix function on time. Equation (6.6) is a matrix version on the KdV equation.

Let $n = 2$

$$u = \begin{bmatrix} \mathcal{A} & \psi \\ \pm \bar{\psi} & \mathcal{A} \end{bmatrix} \quad q = \frac{i\lambda}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (6.7)$$

Now (6.6) is equivalent to following pair of equations.

$$\frac{\partial \mathcal{A}}{\partial t} + c \frac{\partial \mathcal{A}}{\partial x} + 3 \frac{\partial}{\partial x} (\mathcal{A}^2 \pm |\psi|^2) + \frac{\partial^3 \mathcal{A}}{\partial x^3} = 0 \quad (6.8)$$

$$\frac{\partial \psi}{\partial t} + c \frac{\partial \psi}{\partial x} + 6 \frac{\partial}{\partial x} \mathcal{A} \psi + i\lambda \psi + \frac{\partial^3 \psi}{\partial x^3} = 0 \quad (6.9)$$

We will show that system (6.8), (6.9) generates well-known Nonlinear Schrödinger equation Let $k > 0$ – some real number. We choose

$$c = 3k^2 \quad \lambda = -2k^3$$

and introduce new time variable

$$\tau = kt$$

and new unknown function

$$\varphi = \frac{1}{k} e^{-ikx} \psi \quad \psi = k e^{ikx} \varphi \quad (6.10)$$

now system (6.8) (6.9) turns to following

$$\begin{aligned} \frac{\partial}{\partial x} (\mathcal{A} \pm |\varphi|^2) &= -\frac{1}{3k} \frac{\partial \mathcal{A}}{\partial \tau} - \frac{1}{3k^2} \left(\frac{\partial^3 \mathcal{A}}{\partial x^3} + 3 \frac{\partial}{\partial x} \mathcal{A}^2 \right) \\ \frac{\partial \varphi}{\partial \tau} + i \left(\frac{\partial^2 \varphi}{\partial x^2} + 6\mathcal{A}\varphi \right) &+ \frac{1}{k} \left(6 \frac{\partial}{\partial x} \mathcal{A}\varphi + \frac{\partial^3 \varphi}{\partial x^3} \right) = 0 \end{aligned} \quad (6.11)$$

Now let $k \rightarrow \infty$. We set

$$\begin{aligned} \frac{\partial}{\partial x} (\mathcal{A} \pm |\varphi|^2) &= 0 \\ \frac{\partial \varphi}{\partial \tau} + i \left(3 \frac{\partial^2 \varphi}{\partial x^2} + 6\mathcal{A}\varphi \right) &= 0 \end{aligned}$$

using simple scaling $\tau \rightarrow \tau/6$, $x \rightarrow x/\sqrt{2}$ one can get

$$\frac{\partial \varphi}{\partial \tau} + i \left(\frac{\partial^2 \varphi}{\partial x^2} \mp |\varphi|^2 \varphi + \omega(t)\varphi \right) = 0 \quad (6.12)$$

(6.12) is the Nonlinear Schrödinger Equation (NLSE). Linear term $\omega(t)$ can be excluded by multiplication by factor $e^{-i \int_{t_0}^t \omega(t') dt'}$. Equation

$$\frac{\partial \varphi}{\partial \tau} + i \left(\frac{\partial^2 \varphi}{\partial x^2} - |\varphi|^2 \varphi \right) = 0 \quad (6.13)$$

is *defocusing* NLSE. Equation

$$\frac{\partial \varphi}{\partial \tau} + i \left(\frac{\partial^2 \varphi}{\partial x^2} + |\varphi|^2 \varphi \right) = 0 \quad (6.14)$$

is *focusing* NLSE.

Let us use a substitution $\chi \rightarrow \chi e^{-\lambda y}$ in (6.1). In the absence of coefficient dependence on y , the equation turns to

$$\begin{aligned} \hat{\mathbf{L}}\chi &= \lambda\chi, \text{ where } \chi = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} \\ \frac{\partial^2 \chi_1}{\partial x^2} + \mathcal{A}\chi_1 + \psi\chi_2 &= \lambda\chi_1 \\ \frac{\partial^2 \chi_2}{\partial x^2} \pm \bar{\psi}\chi_1 + \mathcal{A}\chi_2 &= \lambda\chi_2 \end{aligned} \quad (6.15)$$

We will assume

$$\begin{aligned} \chi_1 &= \xi_1 e^{\frac{ikx}{2}} & \chi_2 &= \xi_2 e^{-\frac{ikx}{2}} \\ \lambda &= -\frac{k^2}{4} + k\mu \end{aligned} \quad (6.16)$$

Equation (6.15) can be written as follows:

$$\begin{aligned} i\frac{\partial\xi_1}{\partial x} + \phi\xi_2 - \mu\xi_1 &= -\frac{1}{k} \left(\frac{\partial^2\xi_1}{\partial x^2} + \mathcal{A}\xi_1 \right) \\ -i\frac{\partial\xi_2}{\partial x} \pm \bar{\phi}\xi_1 - \mu\xi_2 &= -\frac{1}{k} \left(\frac{\partial^2\xi_2}{\partial x^2} + \mathcal{A}\xi_1 \right) \end{aligned} \quad (6.17)$$

Equation (6.2) in the vector case reads

$$\begin{aligned} k\frac{\partial\chi_1}{\partial\tau} + 4\frac{\partial^3\chi_1}{\partial x^3} + 3(k^2 + 2\mathcal{A})\frac{\partial\chi_1}{\partial x} + 6\psi\frac{\partial\chi_2}{\partial x} \\ + \left(3\frac{\partial\mathcal{A}}{\partial x} - ik^3 \right) \chi_1 + 3\psi_x\chi_2 &= 0 \\ k\frac{\partial\chi_2}{\partial\tau} + 4\frac{\partial^3\chi_2}{\partial x^3} + 3(k^2 + 2\mathcal{A})\frac{\partial\chi_2}{\partial x} \pm 6\bar{\psi}\frac{\partial\chi_1}{\partial x} \\ + \left(3\frac{\partial\mathcal{A}}{\partial x} + ik^3 \right) \chi_2 \pm 3\bar{\psi}_x\chi_1 &= 0 \end{aligned} \quad (6.18)$$

Plugging (6.16) in (6.18) after few simple transformations one gets

$$\begin{aligned} \frac{\partial\xi_1}{\partial\tau} + 6i\frac{\partial^2\xi_1}{\partial x^2} + 6\varphi\frac{\partial\xi_2}{\partial x} + 3i\mathcal{A}\xi_1 + 3\frac{\partial\varphi}{\partial x}\xi_2 &= \\ -\frac{1}{k} \left(4\frac{\partial^3\xi_1}{\partial x^3} + 6\mathcal{A}\frac{\partial\xi_1}{\partial x} + 3\frac{\partial\mathcal{A}}{\partial x}\xi_1 \right) \\ \frac{\partial\xi_2}{\partial\tau} - 6i\frac{\partial^2\xi_2}{\partial x^2} \pm 6\varphi\frac{\partial\xi_1}{\partial x} - 3i\mathcal{A}\xi_2 \pm 3\frac{\partial\bar{\varphi}}{\partial x}\xi_1 &= \\ -\frac{1}{k} \left(4\frac{\partial^3\xi_2}{\partial x^3} + 6\mathcal{A}\frac{\partial\xi_2}{\partial x} + 3\frac{\partial\mathcal{A}}{\partial x}\xi_1 \right) \end{aligned} \quad (6.19)$$

Now one can make the limiting transition $k \rightarrow \infty$ and set the following equation

$$\begin{aligned} i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial\xi}{\partial x} + \begin{bmatrix} 0 & \varphi \\ \pm\bar{\varphi} & 0 \end{bmatrix} \xi &= \mu\xi \\ \frac{\partial\xi}{\partial\tau} + 6i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial^2\xi}{\partial x^2} + 6 \begin{bmatrix} 0 & \varphi \\ \pm\bar{\varphi} & 0 \end{bmatrix} \frac{\partial\xi}{\partial x} \\ + 3 \begin{bmatrix} i\mathcal{A} & \frac{\partial\varphi}{\partial x} \\ \pm\frac{\partial\bar{\varphi}}{\partial x} & i\mathcal{A} \end{bmatrix} \xi &= 0 \\ \xi &= \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \end{aligned} \quad (6.20)$$

Thus for the NLSE

$$\hat{\mathbf{L}} = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} 0 & \varphi \\ \pm\bar{\varphi} & 0 \end{bmatrix} \quad (6.21)$$

for defocusing NLSE it is self-adjoint operator, for focusing NLSE it is non self-adjoint.

$$\hat{\mathbf{A}} = 6i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial^2}{\partial x^2} + 6 \begin{bmatrix} 0 & \varphi \\ \pm\bar{\varphi} & 0 \end{bmatrix} \frac{\partial}{\partial x} + 3 \begin{bmatrix} i\mathcal{A} & \frac{\partial\varphi}{\partial x} \\ \pm\frac{\partial\bar{\varphi}}{\partial x} & -i\mathcal{A} \end{bmatrix} \quad (6.22)$$

Lecture 7

From KP to DS - multiscale expansion

In this chapter we will start with scalar KP2 equation

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} + 3k^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + 3 \frac{\partial}{\partial x} u^2 \right) + 3 \frac{\partial^2 u}{\partial y^2} = 0 \quad (7.1)$$

Here k is some constant. We will not use limiting transition $k \rightarrow \infty$. Instead we will use multiscale expansion.

We present solution of (7.1) in a form

$$u = \frac{\varepsilon}{2} (\Psi e^{i\Phi} + \bar{\Psi} e^{-i\Phi}) + \varepsilon^2 \left(u_0 + \frac{1}{2} \Psi_2 e^{2i\Phi} + \frac{1}{2} \bar{\Psi}_2 e^{-2i\Phi} \right) \quad (7.2)$$

here ε – small parameter, $\Phi = -2k^3 t + kx$

In (7.2) Ψ , u_0 , Ψ_2 are functions of slow variables $\tau = \varepsilon^2 t$, $\xi = \varepsilon x$, $\eta = \varepsilon y$. Plugging (7.2) into (7.1), one can see that terms of order ε are cancelled. After substituting (7.2) to (7.1) we have to cancel terms proportional to $e^{\pm in\Phi}$ separately.

In equation for $n = 0$, $n = 2$ leading order terms are proportional to ε^2 . Putting $n = 0$, one obtains equation

$$\left(k^2 \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) u_0 = -\frac{1}{2} \frac{\partial^2}{\partial \xi^2} |\Psi|^2 \quad (7.3)$$

Amplitude of second harmonic Ψ_2 can be found in explicit form

$$\frac{1}{2} \left[2i \frac{\partial \Phi}{\partial t} + 3k^2 \cdot 2i \frac{\partial \Phi}{\partial x} + \left(2i \frac{\partial \Phi}{\partial x} \right)^3 \right] \Psi_2 + \frac{3 \cdot 2i \Phi_x}{4} \Psi^2 = 0 \quad (7.4)$$

$$\frac{\partial \Phi}{\partial t} = -2k^3 \quad \frac{\partial \Phi}{\partial x} = k$$

From (7.4) one obtains

$$\Psi_2 = \frac{1}{2k^2} \Psi^2 \quad (7.5)$$

In equation for first harmonic $n = 1$ first nonvanishing term is proportional to ε^4 . Collecting all these terms together we obtain the equation

$$ik \frac{\partial \Psi}{\partial \tau} + 3 \left(-k^2 \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \Psi - 6k^2 \left(u_0 \Psi + \frac{1}{2} \Psi_2 \bar{\Psi} \right) = 0$$

taking Ψ_2 from (7.5) we set

$$ik \frac{\partial \Psi}{\partial \tau} + 3 \left(-k^2 \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \Psi - 6k^2 \left(u_0 \Psi + \frac{1}{4k^2} |\Psi|^2 \Psi \right) = 0 \quad (7.6)$$

System (7.3), (7.6) is known as Davey-Stewanson equation.

It is remarkable that sign before second spatial derivative $k^2 \frac{\partial^2}{\partial \xi^2}$ in equation (7.3) and (7.6) are opposite. Transition from KP2 to KP1 equation is performed by change $\frac{\partial}{\partial \eta} \rightarrow i \frac{\partial}{\partial \eta}$, $\frac{\partial^2}{\partial \eta^2} \rightarrow -\frac{\partial^2}{\partial \eta^2}$. If we would start from KP1 equation we should make this replacement. Now the system takes form

$$\left(k^2 \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \eta^2} \right) u_0 = -\frac{1}{2} \frac{\partial^2}{\partial \xi^2} |\Psi|^2$$

$$ik \frac{\partial \Psi}{\partial \tau} - 3 \left(k^2 \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \Psi - 6k^2 \left(u_0 \Psi + \frac{1}{4k^2} |\Psi|^2 \Psi \right) = 0 \quad (7.7)$$

Equation (7.3)-(7.6) have a remarkable special solutions. Suppose that

$$\frac{\partial}{\partial \eta} = \alpha \frac{\partial}{\partial \xi}$$

In other words, a solution depends only on $\xi + \alpha \eta$

Now equation (7.3) can be integrated

$$u_0 = -\frac{1}{2} \frac{1}{k^2 + \alpha^2} |\Psi|^2$$

$$u_0 + \frac{1}{4k^2} |\Psi|^2 = -\frac{k^2 - \alpha^2}{4k^2(k^2 + \alpha^2)} |\Psi|^2$$

Equation (7.6) now takes form

$$ik \frac{\partial \Psi}{\partial \tau} - 3(k^2 - \alpha^2) \left[\frac{\partial^2 \Psi}{\partial \xi^2} - \frac{1}{2} \frac{1}{k^2 + \alpha^2} |\Psi|^2 \Psi \right] = 0 \quad (7.8)$$

For any α it is the *defocusing* Nonlinear Schrödinger equation.

If we start from KP-1 equation, α^2 should be replaced by $-\alpha^2$, and equation (7.8) reads

$$ik \frac{\partial \Psi}{\partial \tau} - 3(k^2 + \alpha^2) \left[\frac{\partial^2 \Psi}{\partial \xi^2} - \frac{1}{2} \frac{1}{k^2 - \alpha^2} |\Psi|^2 \Psi \right] = 0 \quad (7.9)$$

If $\alpha^2 > k^2$, this is *focusing* NLSE.

Now one can address the question — what is going on with Lax pairs in process of multiscale expansion. Equation

$$\frac{\partial \chi}{\partial y} + \hat{\mathbf{L}} \chi = 0$$

$$\frac{\partial \chi}{\partial y} + \frac{\partial^2 \chi}{\partial x^2} + \frac{\varepsilon}{2} (\Psi e^{ikx} + \bar{\Psi} e^{-ikx} + \dots) \chi = 0 \quad (7.10)$$

Equation (7.10) is a *scalar* equation, to turn it to the vector equation, one can find the wave function χ in following form

$$\chi = \left(\chi_1 e^{\frac{ikx}{2}} + \chi_{-1} e^{-\frac{ikx}{2}} + \dots \right) e^{\frac{k^2 y}{4}} \quad (7.11)$$

Equation (7.10) has to be expanded in Fourier series

$$\chi = \sum_{n=1}^{\infty} \chi_n e^{\frac{inkx}{2}}$$

Taking into account only major terms one get following system of equations:

$$\begin{cases} \frac{\partial \chi_1}{\partial y} + ik \frac{\partial \chi_1}{\partial x} + \frac{1}{2} \Psi \chi_{-1} = 0 \\ \frac{\partial \chi_1}{\partial y} - ik \frac{\partial \chi_1}{\partial x} + \frac{1}{2} \bar{\Psi} \chi_1 = 0 \end{cases} \quad (7.12)$$

Equation

$$\frac{\partial \chi}{\partial t} + 4 \frac{\partial^3 \chi}{\partial x^3} + (6u + 3k^2) \frac{\partial \chi}{\partial x} + \left(3 \frac{\partial u}{\partial x} + q \right) \chi = 0 \quad (7.13)$$

$$\frac{\partial q}{\partial x} = -3 \frac{\partial u}{\partial y}$$

also can be transformed to a system of equations for χ_1, χ_{-1} . To do this, we again present χ in form (7.11) and use expansion (7.2). We realize that first non-vanishing terms are of order ε^2 . Collecting them together one see that equation (7.13) is now a system

$$\frac{\partial \chi_1}{\partial \tau} + 6ik \frac{\partial^2 \chi_1}{\partial \xi^2} + 3\Psi \frac{\partial \chi_{-1}}{\partial \xi} + 3iku_0 \chi_1 + \frac{3}{2} \frac{\partial \Psi}{\partial \xi} \chi_{-1} = 0$$

$$\frac{\partial \chi_{-1}}{\partial \tau} - 6ik \frac{\partial^2 \chi_{-1}}{\partial \xi^2} + 3\bar{\Psi} \frac{\partial \chi_1}{\partial \xi} - 3iku_0 \chi_{-1} + \frac{3}{2} \frac{\partial \bar{\Psi}}{\partial \xi} \chi_1 = 0 \quad (7.14)$$

$$(7.15)$$

$$\hat{\mathbf{L}} = ik \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial \xi} + \frac{1}{2} \begin{pmatrix} 0 & \Psi \\ \bar{\Psi} & 0 \end{pmatrix}$$

$$\hat{\mathbf{A}} = 6ik \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial^2}{\partial \xi^2} + 3 \begin{pmatrix} 0 & \Psi \\ \bar{\Psi} & 0 \end{pmatrix} \frac{\partial}{\partial \xi} + \frac{3}{2} \begin{pmatrix} 2iku_0 & \frac{\partial \Psi}{\partial \xi} \\ \frac{\partial \bar{\Psi}}{\partial \xi} & -2iku_0 \end{pmatrix} \quad (7.16)$$

Lecture 8

Hamiltonian formalism for waves in weakly nonlinear media

We study a medium described by one pair of canonical variables ρ, Φ

$$\frac{\partial \rho}{\partial t} = \frac{\delta H}{\delta \Phi} \quad \frac{\partial \Phi}{\partial t} = -\frac{\delta H}{\delta \rho} \quad (8.1)$$

The Hamiltonian H is some functional on ρ, Φ . Let us perform Fourier transform

$$\begin{aligned} \Phi(\vec{r}) &= \frac{1}{(2\pi)^{d/2}} \int \Phi(\vec{k}) e^{i\vec{k}\vec{r}} d\vec{k}; & \rho(\vec{r}) &= \frac{1}{(2\pi)^{d/2}} \int \rho(\vec{k}) e^{i\vec{k}\vec{r}} d\vec{k} \\ \Phi(\vec{k}) &= \frac{1}{(2\pi)^{d/2}} \int \Phi(\vec{r}) e^{-i\vec{k}\vec{r}} d\vec{r}; & \rho(\vec{k}) &= \frac{1}{(2\pi)^{d/2}} \int \rho(\vec{r}) e^{-i\vec{k}\vec{r}} d\vec{r} \end{aligned} \quad (8.2)$$

From now on during the next three lectures we will omit the vector-sign of \vec{k}, \vec{q} and \vec{r} . After Fourier transform

$$\begin{aligned} \frac{\partial \rho(k)}{\partial t} &= \frac{1}{(2\pi)^{d/2}} \int e^{-ikr} \frac{\delta H}{\delta \Phi} dr \\ \frac{\delta H}{\delta \Phi(r)} &= \int \frac{\delta H}{\delta \Phi(q)} \frac{\delta \Phi(q)}{\delta \Phi(r)} dq \end{aligned}$$

$$\frac{\delta\Phi(q)}{\delta\Phi(r)} = \frac{1}{(2\pi)^{d/2}} e^{-iqr} \quad (8.3)$$

Hence

$$\frac{\delta H}{\delta\Phi(r)} = \frac{1}{(2\pi)^{d/2}} \int \frac{\delta H}{\delta\Phi(q)} e^{-iqr} dq$$

Thus

$$\frac{\partial\rho(k)}{\partial t} = \frac{1}{(2\pi)^d} \int e^{-i(k+q,r)} \frac{\delta H}{\delta\Phi(q)} dr dq$$

However

$$\frac{1}{(2\pi)^d} \int e^{-i(k+q,r)} dr = \delta(k+q) \quad (8.4)$$

Finally

$$\frac{\partial\rho(k)}{\partial t} = \frac{\delta H}{\delta\Phi(-k)}$$

But $\Phi(-k) = \bar{\Phi}(k)$, $\rho(-k) = \bar{\rho}(k)$ After Fourier transform equation (8.1) take from

$$\frac{\partial\rho}{\partial t} = \frac{\delta H}{\delta\bar{\Phi}(k)} \quad \frac{\partial\Phi(k)}{\partial t} = -\frac{\delta H}{\delta\bar{\rho}(k)} \quad (8.5)$$

For beginning we suppose that the Hamiltonian H is quadratic functional invariant with respect to spatial transitions. The most general form of such functional is following

$$\begin{aligned} H = & \frac{1}{2} \int A(k)\Phi(k)\bar{\Phi}(k)dk + \frac{1}{2} \int B(k)\rho(k)\bar{\Phi}(k)dk + \\ & + \frac{1}{2} \int B'(k)\bar{\rho}(k)\Phi(k)dk + \frac{1}{2} \int C(k)\rho(k)\bar{\rho}(k)dk \end{aligned} \quad (8.6)$$

H is a real functional, hence

$$\begin{aligned} A(-k) &= A(k) = \bar{A}(k) \\ C(-k) &= C(k) = \bar{C}(k) \\ B(k) &= \bar{B}'(k) \end{aligned} \quad (8.7)$$

$B(-k) = B'(k)$ this means

$$\begin{aligned} B(k) &= B_1 + iB_2(k) \\ B_1(-k) &= B_1(k) \\ B_2(-k) &= -B_2(k) \end{aligned} \quad (8.8)$$

Equations (8.5) now are

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= A\Phi + (B_1 + iB_2)\rho \\ \frac{\partial \Phi}{\partial t} &= (-B_1 + iB_2)\Phi - C\rho \end{aligned} \quad (8.9)$$

Assuming that $\rho, \Phi \sim e^{i\omega t}$ one obtains

$$\omega(k) = B_2 \pm \Delta \quad \Delta = \sqrt{AC - B_1^2} \quad (8.10)$$

The medium is stable if $AC - B_1^2 > 0$. In virtue of (8.7) (8.8)

$$\Delta(-k) = \Delta(k)$$

Thus $\Delta(k)$ and $B_2(k)$ are symmetric and antisymmetric parts of frequency. Equation for $\omega(k)$ has two solutions, but only one of them has physical meaning. To determine the sign in (8.10), we have to introduce *normal variables*

$$a(k) = u(k)\rho(k) + v(k)\Phi(k) \quad (8.11)$$

Such that

$$\frac{\partial a(k)}{\partial t} = i\omega(k)a(k) \quad (8.12)$$

sign in $\omega(k)$ is still not known. Plugging (8.11) to (8.12) and using equation (8.9) one gets

$$\begin{aligned} (i\omega + (\bar{B}))v &= uA \\ u &= \frac{1}{A}(B_1 \pm i\Delta)v \end{aligned}$$

we assume $v(-k) = v(k) = \bar{v}(k)$. Thus

$$\begin{aligned} a(k) &= v \left[\Phi + \frac{1}{A}(B_1 \pm i\Delta)\rho \right] \\ a^*(-k) &= v \left[\Phi + \frac{1}{A}(B_1 \mp i\Delta)\rho \right] \end{aligned} \quad (8.13)$$

Then

$$\rho = \mp \frac{iA}{2v\Delta} (a(k) - a^*(-k)) \quad (8.14)$$

$$\begin{aligned} \Phi &= \frac{1}{2v} (a(k) + a^*(k)) - \frac{B_1}{A} \rho = \\ &= \frac{1}{2v} \left[\left(1 \pm \frac{iB_1}{\Delta} \right) a(k) + \left(1 \mp \frac{iB_1}{\Delta} \right) a^*(-k) \right] \end{aligned} \quad (8.15)$$

To determine $v(k)$ we substitute (8.15), (8.14) to the Hamiltonian and demand that

$$H = \int \omega(k) a(k) \bar{a}(k) dk \quad (8.16)$$

A relatively long calculation leads to a very simple result

$$v^2 = \pm \frac{A}{2\Delta} \quad (8.17)$$

or

$$v = \frac{|A|^{\frac{1}{2}}}{\sqrt{2\Delta}} \quad (8.18)$$

obviously sign in (8.17) should coincide with the sign of A . Finally

$$\omega(k) = B_2(k) + \text{sign}A(k) \cdot \Delta(k)$$

Sign of $\omega(k)$ has clear physical meaning – Waves with wave vector \vec{k} have *positive energy* if $\omega(k) > 0$ and negative energy in the opposite case.

For normal variable one gets

$$\begin{aligned} \rho &= -i \text{sign}A \frac{|A|^{1/2}}{\sqrt{2}\Delta^{1/2}} (a(k) - a^*(-k)) \\ \Phi &= \frac{\Delta^{1/2}}{\sqrt{2|A|}} \left[\left(1 \pm \frac{iB_1}{\Delta} \right) a(k) + \left(1 \mp \frac{iB_1}{\Delta} \right) a^*(-k) \right] \end{aligned} \quad (8.19)$$

Comment

In nonlinear media the Hamiltonian cannot be transformed to form (8.16).

In a weakly nonlinear medium the Hamiltonian can be expanded in terms of canonical variables.

$$H = H_2 + H_3 + H_4 \dots \quad (8.20)$$

One can perform this expansion in normal variables. Now

$$\begin{aligned}
H_2 &= \int \omega(k) a(k) a^*(k) dk \\
H_3 &= H_3^{(1)} + H_3^{(2)} \\
H_3^{(1)} &= \frac{1}{3!} \int \left\{ V_{k,k_1,k_2}^{(0,3)} a_k a_{k_1} a_{k_2} + \text{c.c.} \right\} \delta(k + k_1 + k_2) dk dk_1 dk_2 \quad (8.21) \\
H_3^{(2)} &= \frac{1}{1!2!} \int \left\{ V_{k,k_1,k_2}^{(1,2)} a_k^* a_{k_1} a_{k_2} + \text{c.c.} \right\} \delta(k - k_1 - k_2) dk dk_1 dk_2 = H_{1,2}
\end{aligned}$$

The whole Hamiltonian can be presented as a sum

$$\begin{aligned}
H &= \sum_{n,m} H_{n,m} \quad n \leq m \\
H_{n,m} &= \frac{1}{n!m!} \left\{ V_{k_1 \dots k_n, k_{n+1} \dots k_{n+m}} a^*(k_1) \dots a^*(k_n) a(k_{n+1}) \dots a(k_{n+m}) + \text{c.c.} \right\} \times \\
&\quad \times \delta(k_1 + \dots + k_n - k_{n+1} - \dots - k_{n+m}) dk_1 \dots dk_{n+m}
\end{aligned}$$

Lecture 9

Three waves interactions

In the last chapter we introduced normal variable a_k

$$a_k = u_k \rho_k + v_k \Phi_k \quad (9.1)$$

and defined coefficients u_k, v_k from two conditions:

1. a_k satisfies the equations

$$\frac{\partial a_k}{\partial t} = i\omega_k a_k \quad (9.2)$$

2. Quadratic Hamiltonian H_2 (8.16) has diagonal form

$$H_2 = \int \omega_k a_k a_k^* dk$$

Now equation (9.2) can be presented as follows

$$\frac{\partial a_k}{\partial t} = i \frac{\delta H}{\delta a_k^*} \quad (9.3)$$

Now we address following question. Let H is an arbitrary Hamiltonian. What condition we have to impose on u, v to provide validity of equation (9.3)?

From (9.1) one obtains

$$\begin{aligned}
a_k^* &= u_k^* \rho_k^* + v_k^* \Phi_k^* \\
a_{-k} &= u_{-k} \rho_k^* + v_{-k} \Phi_k^*
\end{aligned} \tag{9.4}$$

Then, after differentiation of (9.3) by time

$$\begin{aligned}
\frac{\partial a_k}{\partial t} &= u_k \frac{\partial \rho_k}{\partial t} + v_k \frac{\partial \Phi_k}{\partial t} \\
i \frac{\delta H}{\delta a_k^*} &= u_k \frac{\delta H}{\delta \Phi_k^*} - v_k \frac{\delta H}{\delta \rho_k^*}
\end{aligned} \tag{9.5}$$

$$\frac{\delta H}{\delta \rho_k^*} = \frac{\delta H}{\delta a_k^*} \frac{\delta a_k^*}{\delta \rho_k^*} + \frac{\delta H}{\delta a_{-k}} \frac{\delta a_{-k}}{\delta \rho_k^*} = u_k^* \frac{\delta H}{\delta a_k^*} + u_{-k} \frac{\delta H}{\delta a_{-k}} \tag{9.6}$$

In the same way

$$\frac{\delta H}{\delta \Phi_k^*} = v_k^* \frac{\delta H}{\delta a_k^*} + v_{-k} \frac{\delta H}{\delta a_{-k}} \tag{9.7}$$

Substituting (9.6), (9.7) to (9.5) and assuming that condition (9.3) is satisfied for *any* Hamiltonian H , one gets

$$i \frac{\delta H}{\delta a_k^*} = u_k (v_k^* \frac{\delta H}{\delta a_k^*} + v_{-k} \frac{\delta H}{\delta a_{-k}}) - v_k (u_k^* \frac{\delta H}{\delta a_k^*} + u_{-k} \frac{\delta H}{\delta a_{-k}}) \tag{9.8}$$

Condition (9.8) must be valid for any Hamiltonian H . It implies following condition on coefficients

$$u_k v_{-k} = u_{-k} v_k \tag{9.9}$$

$$u_k v_k^* - v_k u_k^* = i \tag{9.10}$$

In previous chapter we found

$$u_k = \frac{1}{A} (B_1 \pm i\Delta) v_k \quad v_k^2 = \pm \frac{A}{2\Delta} \tag{9.11}$$

Substituting (9.11) into (9.9), (9.10) one can see that these conditions are satisfied.

We have proved a very important theorem - change of variables (8.19) transforms any Hamiltonian equation (8.1) to equation (9.3).

Suppose that Hamiltonian H is expanded in series in powers of $a_k a_k^*$. We assume that the medium is homogeneous. It means that Hamiltonian should be invariant with respect to transform $a_k \rightarrow a_k e^{i(k, \lambda)}$ where λ -arbitrary vector. The Hamiltonian is

$$H = H_2 + H_3 + \dots$$

$$H_2 = \int \omega_k |a_k|^2 dk$$

$$H_3 = H_3^{(1)} + H_3^{(2)}$$

$$H_3^{(1)} = \frac{1}{6} \int \{V_{k, k_1, k_2}^{(0,3)} a_k a_{k_1} a_{k_2} + V_{k, k_1, k_2}^{*(0,3)} a_k^* a_{k_1}^* a_{k_2}^*\} \delta(k - k_1 - k_2) dk dk_1 dk_2 \quad (9.12)$$

$$H_3^{(2)} = \frac{1}{2} \int \{V_{k, k_1, k_2}^{(1,2)} a_k^* a_{k_1} a_{k_2} + V_{k, k_1, k_2}^{*(1,2)} a_k a_{k_1}^* a_{k_2}^*\} \delta(k - k_1 - k_2) dk dk_1 dk_2 \quad (9.13)$$

Equations (9.3) are following

$$\begin{aligned} \frac{\partial a_k}{\partial t} = & i\omega_k a_k + i \int \left\{ \frac{1}{2} V_{k, k_1, k_2}^{*(0,3)} a_{k_1}^* a_{k_2}^* \delta(k + k_1 + k_2) + \right. \\ & \left. + \frac{1}{2} V_{k, k_1, k_2}^{(1,2)} a_{k_1} a_{k_2} \delta(k - k_1 - k_2) + V_{k, k_1, k_2}^{*(1,2)} a_{k_1}^* a_{k_2}^* \delta(k + k_1 - k_2) \right\} dk_1 dk_2 \quad (9.14) \end{aligned}$$

Note that in a general case the Hamiltonian H can be presented as follows

$$H = \sum_{n, m} H_{n, m} \quad n \leq m$$

$$\begin{aligned} H_{nm} = \frac{1}{n! m!} \int \{ & V_{k_1, \dots, k_n, k_{n+1}, \dots, k_{n+m}}^{(n, m)} a_{k_1} \dots a_{k_n} a_{k_{n+1}}^* \dots a_{k_{n+m}}^* \delta(k_1 + \\ & + \dots k_n - k_{n+1} \dots - k_{n+m}) + \dots \} dk_1 \dots dk_{n+m} \quad (9.15) \end{aligned}$$

Suppose now that $\omega(k) \geq 0$ and

$$\omega(k_1 + k_2) = \omega(k_1) + \omega(k_2) \quad (9.16)$$

has nontrivial solutions. In the simplest case, when dimension of the space $d \geq 2$ and $\omega'(k) > 0$ $\omega = \omega(|k|)$ one can find a sufficient condition for solvability of equation (9.16). This condition is positivity of dispersion $\omega'' > 0$. Indeed, in this case, for parallel vectors k_1, k_2

$$\omega(k_1 + k_2) > \omega(k_1) + \omega(k_2) \quad (9.17)$$

Now replacing $k_1 \rightarrow k_1 + k_\perp$, $k_2 \rightarrow k_2 - k_\perp$ where $(k_\perp, k_1) = 0$, one can turn inequality (9.17) to equality.

Suppose we found three wave vectors p_1, p_2, p_3 comprising the resonant triad

$$\begin{aligned} p_1 &= p_2 + p_3 \\ \omega(p_1) &= \omega(p_2) + \omega(p_3) \end{aligned} \quad (9.18)$$

Let the wave filed $a(k)$ is a combination of three quasi-monochromatic wave trains

$$a(k) = A(k) + B(k) + C(k) \quad (9.19)$$

We assume that $A(k) = 0$ if $|k - p_1| > \epsilon$, $B(k) = 0$ if $|k - p_2| > \epsilon$, $C(k) = 0$ if $|k - p_3| > \epsilon$. Here ϵ is a small parameter. Functions A, B, C are concentrated near wave vectors p_1, p_2, p_3 .

To construct approximate envelope equations we perform change of variables

$$a_k = e^{i\omega_k t} c_k \quad (9.20)$$

From (9.14) we obtain

$$\begin{aligned} \frac{\partial c_k}{\partial t} &= i \int \left\{ \frac{1}{2} V_{k,k_1,k_2}^{*(0,3)} c_{k_1}^* c_{k_2}^* e^{-i(\omega_k + \omega_{k_1} + \omega_{k_2})t} \delta(k + k_1 + k_2) + \right. \\ &\quad \left. + \frac{1}{2} v_{k,k_1,k_2}^{(1,2)} c_{k_1} c_{k_2} e^{-i(\omega_k - \omega_{k_1} - \omega_{k_2})t} \delta(k - k_1 - k_2) + \right. \end{aligned} \quad (9.21)$$

$$\left. + V_{k,k_1,k_2}^{*(1,2)} c_{k_1}^* c_{k_2} e^{-i(\omega_k + \omega_{k_1} - \omega_{k_2})t} \delta(k + k_1 - k_2) \right\} dk_1 dk_2 \quad (9.22)$$

Let in (9.20) k is close to p_1 , $\omega(k)$ -close to $\omega(p_1)$. In the right hand side of (9.21) most terms are fast oscillating. In the limit of small amplitude $|a_k| \rightarrow 0$ they can be neglected. Only ‘‘secular’’, non-oscillating terms are important. Looking at the right hand side of equation (9.21), one can see that only second term can be slowly oscillating, if we assume $k_1 \simeq p_2$, $k_2 \simeq p_3$ or viceversa. Taking into account only these terms and returning back from c_k to a_k one can put approximately

$$\frac{\partial A}{\partial t} = i\omega(k)A_k + i \int V_{k,k_1,k_2}^{(1,2)} B_{k_1} C_{k_2} \delta_{k-k_1-k_2} dk_1 dk_2 \quad (9.23)$$

Now one can put

$$k = p_1 + \varkappa \quad k_1 = p_2 + \varkappa_1 \quad k_2 = p_3 + \varkappa_2$$

and introduce

$$A = \tilde{A} e^{i\omega(p_1)t} \quad B = \tilde{B} e^{i\omega(p_2)t} \quad C = \tilde{C} e^{i\omega(p_3)t}$$

Then equation (9.23) reads

$$\frac{\partial \tilde{A}_\varkappa}{\partial t} = i(\omega(p_1 + \varkappa) - \omega(p_1)) \tilde{A}_\varkappa + i \int V_{p_1+\varkappa, p_2+\varkappa_1, p_3+\varkappa_2}^{(1,2)} \tilde{B}_{\varkappa_1} \tilde{C}_{\varkappa_2} \delta_{\varkappa-\varkappa_1-\varkappa_2} d\varkappa_1 d\varkappa_2 \quad (9.24)$$

Now we can put approximately

$$\omega(p_1 + \varkappa) - \omega(p_1) = (v, \varkappa)$$

here $v_1 = \frac{\partial \omega}{\partial k}|_{k=p_1}$ -group velocity of the first wave train $A(k)$ and

$$V_{p_1+\varkappa, p_2+\varkappa_1, p_3+\varkappa_2}^{(1,2)} \simeq (2\pi)^{-d/2} U_{p_1, p_2, p_3}$$

U is a constant.

Now from (9.24) one obtains

$$\frac{\tilde{A}}{\delta t} = i(\varkappa v_1) \tilde{A}_\varkappa + i(2\pi)^{-d/2} U \int \tilde{B}_{\varkappa_1} \tilde{C}_{\varkappa_2} \delta(\varkappa - \varkappa_1 - \varkappa_2) d\varkappa_1 d\varkappa_2$$

Performing Fourier transform

$$a(r) = \frac{1}{(2\pi)^{d/2}} \int \tilde{A}(\boldsymbol{\varkappa}) e^{i\boldsymbol{\varkappa}r} d\boldsymbol{\varkappa}$$

$$\tilde{A}(\boldsymbol{\varkappa}) = \frac{1}{(2\pi)^{d/2}} \int a(r) e^{-i\boldsymbol{\varkappa}r} dr$$

and similar transforms for \tilde{B} , \tilde{C} , one ends up with following PDE

$$\frac{\partial a}{\partial t} + (v_1 \nabla) a = i u b c$$

Repeating this procedure, assuming $k \simeq p_2$ and $k \simeq p_3$, we get finally following system of nonlinear PDE

$$\begin{aligned} \frac{\partial a}{\partial t} + (v_1 \nabla) a &= i u b c \\ \frac{\partial b}{\partial t} + (v_2 \nabla) b &= i u^* a c^* \\ \frac{\partial c}{\partial t} + (v_3 \nabla) c &= i u^* a b^* \end{aligned} \tag{9.25}$$

here

$$v_2 = \left. \frac{\partial \omega}{\partial k} \right|_{k=p_2} \quad v_3 = \left. \frac{\partial \omega}{\partial k} \right|_{k=p_3}$$

In these equations u is a complex constant.

$$u = (2\pi)^{d/2} V_{p_1, p_2, p_3}^{(1,2)} = |u| e^{i\phi}$$

By simple transform

$$a \rightarrow |u|^{-1/2} e^{-i\phi} a \quad b \rightarrow |u|^{-1/2} e^{-i\phi} b \quad c \rightarrow |u|^{-1/2} e^{-i\phi} c$$

one can put $u = 1$ and finally get the system

$$\begin{aligned} \frac{\partial a}{\partial t} + (v_1 \nabla) a &= i b c \\ \frac{\partial b}{\partial t} + (v_2 \nabla) b &= i a c^* \\ \frac{\partial c}{\partial t} + (v_3 \nabla) c &= i a b^* \end{aligned} \tag{9.26}$$

This is so-called “three-wave system”. This system admits following reduction

$$a = iu \quad b = iv \quad c = iw$$

where u, v, w -are real functions. Now

$$\begin{aligned} \frac{\partial u}{\partial t} + (v_1 \nabla)u &= -vw \\ \frac{\partial v}{\partial t} + (v_2 \nabla)v &= uw \\ \frac{\partial w}{\partial t} + (v_3 \nabla)w &= uv \end{aligned} \tag{9.27}$$

In the spatial homogenous case this system simplifies to the system of ODE's

$$\begin{aligned} \frac{\partial u}{\partial t} &= -vw \\ \frac{\partial v}{\partial t} &= uw \\ \frac{\partial w}{\partial t} &= uv \end{aligned} \tag{9.28}$$

identical to the Euler equation for free rotation of three-dimensional rigid body.

Spatially homogenous system (9.26) reads

$$\begin{aligned} \frac{\partial a}{\partial t} &= ibc \\ \frac{\partial b}{\partial t} &= iac^* \\ \frac{\partial c}{\partial t} &= iab^* \end{aligned} \tag{9.29}$$

This system has following constant of motion

$$\begin{aligned}
|a|^2 + |b|^2 &= I_1 \\
|a|^2 + |c|^2 &= I_2
\end{aligned}
\tag{9.30}$$

In the case of amplitudes tending to zero at boundaries (or in infinity) general system (9.26) has constants of motion

$$\begin{aligned}
\int |a|^2 dr + \int |b|^2 dr &= I_1 \\
\int |a|^2 dr + \int |c|^2 dr &= I_2
\end{aligned}
\tag{9.31}$$

These motion constants are known as Manley-Row relations.

Now suppose that $\omega(k)$ can change sign. In other words, we have both waves of positive and negative energy. In such medium one could find a triad satisfying following conditions

$$\begin{aligned}
p_1 + p_2 + p_3 &= 0 \\
\omega(p_1) + \omega(p_2) + \omega(p_3) &= 0
\end{aligned}
\tag{9.32}$$

For these triads the main nonlinear term is $H_3^{(1)}$. Repeating previous considerations one ends up with the system

$$\begin{aligned}
\frac{\partial a}{\partial t} + (v_1 \nabla) a &= -ib^* c^* \\
\frac{\partial b}{\partial t} + (v_2 \nabla) b &= -ia^* c^* \\
\frac{\partial c}{\partial t} + (v_3 \nabla) c &= -ia^* b^*
\end{aligned}
\tag{9.33}$$

The Manley-Row relations now take form

$$\begin{aligned}
\int |a|^2 dr - \int |b|^2 dr &= I_1 \\
\int |a|^2 dr - \int |c|^2 dr &= I_2
\end{aligned}
\tag{9.34}$$

System (9.33) has a simple solution

$$\begin{aligned}
a = b = c &= -iu \\
\frac{\partial u}{\partial t} &= u^2 \\
u &= \frac{1}{t_0 - t}
\end{aligned}
\tag{9.35}$$

This equation describes spatially uniform explosion (collapse).

Lecture 10

From KP-1 Equation to 3-wave System

In this lecture we will derive the 3-wave system from the KP-1 equation and find Lax pair for 3-wave system. We start from the KP-1 equation

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} + 3 \frac{\partial u^2}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right) - 3 \frac{\partial^2 u}{\partial y^2} = 0 \quad (10.1)$$

After Fourier transform

$$u(x, y) = \frac{1}{2\pi} \int u(p, q) e^{i(px+qy)} dp dq$$

p, q are components of wave vector $k = (p, q)$ and

$$u(p, q) = \frac{1}{2\pi} \int u(x, y) e^{-i(px+qy)} dx dy$$

Equation (10.1) takes form

$$\frac{\partial u_k}{\partial t} = i\omega(k)u_k - \frac{3ip}{2\pi} \int u_{k_1} u_{k_2} \delta(k - k_1 - k_2) dk_1 dk_2 \quad (10.2)$$

Let in (10.2) $p > 0$

As far as

$$u_{-k} = u_k^* \quad (10.3)$$

one can transform equation (10.2) to following form

$$\begin{aligned} \frac{\partial u(p, q)}{\partial t} = i\omega(p, q)u(p, q) - \frac{3ip}{2\pi} \int_{\substack{p_1 > 0 \\ p_2 > 0}} (u_{p_1 q_1} u_{p_2 q_2} \delta(p - p_1 - p_2) \delta(q - q_1 - q_2) + \\ + 2u_{p_1 q_1}^* u_{p_2 q_2} \delta(p + p_1 + p_2) \delta(q + q_1 - q_2)) dp_1 dp_2 dq_1 dq_2 \end{aligned}$$

Here

$$\omega(p, q) = p^3 + \frac{3q^2}{p} \quad (10.4)$$

Now we introduce normal variables

$$u_{p, q} = \sqrt{\frac{p}{2}} a_{p, q}$$

In these variables

$$\begin{aligned} \frac{\partial a(p, q)}{\partial t} = i\omega(p, q)a(p, q) - \frac{3ip}{2\pi} \int_{\substack{p_1 > 0 \\ p_2 > 0}} (pp_1 p_2)^{1/2} (u_{p_1 q_1} u_{p_2 q_2} \delta(p - p_1 - p_2) \delta(q - q_1 - q_2) + \\ + 2u_{p_1 q_1}^* u_{p_2 q_2} \delta(p + p_1 + p_2) \delta(q + q_1 - q_2)) dp_1 dp_2 dq_1 dq_2 \end{aligned} \quad (10.5)$$

One can check directly that equation (10.5) can be written in the form

$$\begin{aligned} \frac{\partial a(p, q)}{\partial t} = i \frac{\delta H}{\delta a_{p, q}^*} \\ H = \int_{p > 0} \omega(p, q) |a_{pq}|^2 dp dq - \\ \frac{3}{4\pi} \int_{\substack{p_1 > 0 \\ p_2 > 0 \\ p_3 > 0}} (p_1 p_2 p_3)^{1/2} (u_{p_1 q_1}^* u_{p_2 q_2} u_{p_3 q_3} + u_{p_1 q_1} u_{p_2 q_2}^* u_{p_3 q_3}^*) \times \\ \times \delta(p_1 - p_2 - p_3) \delta(q_1 - q_2 - q_3) dp_1 dq_1 dp_2 dq_2 dp_3 dq_3 \end{aligned} \quad (10.6)$$

One can easily check that in initial variables the Hamiltonian is

$$H = \frac{1}{2} \int \left\{ \left[u_x^2 + 3 (\partial^{-1} u_y)^2 \right] - u^3 \right\} dx dy \quad (10.7)$$

Later on we will show that equation (10.1) has infinite number of motion constants. They will be discussed in the further lectures. So far we established that KP-1 equation is a Hamiltonian system belonging to the class described in lecture 9.

This could be starting point for derivation of the three-wave system. First of all we have to be sure that resonant triads do exist. To see this we note first that dispersion relation (10.4) can be parameterized as follows

$$\begin{aligned} p &= \xi_1 - \xi_2 \\ q &= \pm(\xi_1^2 - \xi_2^2) \\ \omega &= 4(\xi_1^3 - \xi_2^3) \end{aligned} \quad (10.8)$$

As far as $p > 0$ $\xi_1 > \xi_2$. Since this time, we will study only the case $\xi_1^2 > \xi_2^2$, $q > 0$ and ξ_1, ξ_2 are positive. Let us consider the triad

$$\begin{aligned} p_1 &= \xi_1 - \xi_2 & p_2 &= \xi_2 - \xi_3 & p_3 &= \xi_1 - \xi_3 \\ q_1 &= \xi_1^2 - \xi_2^2 & q_2 &= \xi_2^2 - \xi_3^2 & q_3 &= \xi_1^2 - \xi_3^2 \\ \omega_1 &= 4(\xi_1^3 - \xi_2^3) & \omega_2 &= 4(\xi_2^3 - \xi_3^3) & \omega_3 &= 4(\xi_1^3 - \xi_3^3) \end{aligned} \quad (10.9)$$

Apparently (10.9) comprise the resonant triad

$$\begin{aligned} p_1 + p_2 &= p_3 \\ q_1 + q_2 &= q_3 \\ \omega_1 + \omega_2 &= \omega_3 \end{aligned}$$

Now we will derive a system, more general then 3-wave.

Let us define $\xi_1 > \xi_2 > \dots > \xi_n > 0$

$$\begin{aligned} \Phi_{ik} &= (\xi_i - \xi_k)x + (\xi_i^2 - \xi_k^2)y + 4(\xi_i^3 - \xi_k^3)t \\ \Phi_{ik} + \Phi_{kj} &= \Phi_{ij} \quad \Phi_{ki} = -\Phi_{ik} \end{aligned} \quad (10.10)$$

Suppose that the wave field u is a composition of quasi-monochromatic wave trains

$$\begin{aligned} u &= \sum_{i,j} \varepsilon u_{ij}(\xi, \eta) e^{i\Phi_{ij}} \quad i \neq j \\ u_{ji} &= u_{ij}^* \\ \xi &= \varepsilon x \quad \eta = \varepsilon y \quad \tau = \varepsilon t \end{aligned} \quad (10.11)$$

Here ε is a small parameter. Let us look for $e^{-i\Phi_{ij}} u^2$. It has oscillating and non-oscillating terms. Neglecting oscillating terms, we set

$$e^{-i\Phi_{ij}} u^2 \simeq \sum_{\substack{k \neq i \\ k \neq j}} \varepsilon^2 u_{ik} u_{kj} + \dots \quad (10.12)$$

Plugging (10.11) in (10.1) and keeping only terms of order ε^2 , one obtains the following system

$$\frac{\partial}{\partial \tau} u_{ij} = (\vec{v}_{ij} \nabla) u_{ij} - 3i(\xi_i - \xi_j) \sum_{\substack{k \neq i \\ k \neq j}} u_{ik} u_{kj} \quad (10.13)$$

$$\begin{aligned} \vec{v}_{ij} \nabla &= v_{ij}^{(1)} \frac{\partial}{\partial \xi} + v_{ij}^{(2)} \frac{\partial}{\partial \eta} \\ v_{ij}^{(1)} &= \frac{\partial}{\partial p} \omega(p, q) = 3 \left(p^2 - \frac{q^2}{p^2} \right) = 12\xi_i \xi_j \\ v_{ij}^{(2)} &= \frac{\partial}{\partial q} \omega(p, q) = \frac{6q}{p} = 6(\xi_1 + \xi_2) \end{aligned} \quad (10.14)$$

In the simplest case $n = 3$, assuming $u_{13} = (\xi_1 - \xi_3)^{1/2} A$, $u_{12} = (\xi_1 - \xi_3)^{1/2} B$, $u_{23} = (\xi_2 - \xi_3)^{1/2} C$ one obtains 3-wave system with interaction coefficient

$$u = -3(\xi_1 - \xi_2)^{1/2} (\xi_1 - \xi_3)^{1/2} (\xi_2 - \xi_3)^{1/2}$$

To find Lax pair for system (10.13) we start with the first Lax operator for KP-1

$$-i \frac{\partial \chi}{\partial y} + \frac{\partial^2 \chi}{\partial x^2} + u \chi = 0 \quad (10.15)$$

we will seek its solutions in a form

$$\begin{aligned} \chi &= \sum_{k=1}^n \chi_k e^{i\xi_k x + i\xi_k^2 y + 4i\xi_k^3 t} \\ \chi_k &= \chi_k(\xi, \eta, \tau) \end{aligned} \quad (10.16)$$

Keeping in (10.15) only non-oscillating terms of order ε , one obtains the system

$$i \left(-\frac{\partial \chi_j}{\partial \eta} + 2\xi_j \frac{\partial \chi_j}{\partial \xi} \right) + \sum_{k \neq j} u_{jk} \chi_k = 0 \quad (10.17)$$

Repeating this procedure with the second equation one yields

$$\frac{\partial \chi_i}{\partial \tau} - 12\xi_i^2 \frac{\partial \chi_i}{\partial \xi} + 3i \sum_{k=1}^n (\xi_i + \xi_k) u_{ik} \chi_k = 0 \quad (10.18)$$

One can check that the compatibility conditions for equations (10.17), (10.18) is exactly equation (10.10)

System (10.13) is a special class of more general system of n-waves. To construct the general system we start with the following overdetermined pair of linear equations

$$\frac{\partial \Psi}{\partial y} = F \frac{\partial \Psi}{\partial x} + [F, Q] \Psi \quad (10.19)$$

$$\frac{\partial \Psi}{\partial t} = G \frac{\partial \Psi}{\partial x} + [G, Q] \Psi \quad (10.20)$$

Here F, G are commuting matrices: $[F, G] = 0$. One can think that they are diagonal:

$$F = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} \quad G = \begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_n \end{bmatrix}$$

Compatibility condition for system (10.19), (10.20) is following

$$\frac{\partial}{\partial t}[F, Q] - \frac{\partial}{\partial y}[F, Q] + F \frac{\partial}{\partial x}[G, Q] - G \frac{\partial}{\partial x}[F, Q] - [[F, Q], [G, Q]] = 0 \quad (10.21)$$

One can consider that diagonal elements of Q are zero. Then equation (10.21) read

$$(b_i - b_j) \frac{\partial Q_{ij}}{\partial y} + (b_i a_j - a_j b_i) \frac{\partial Q_{i,j}}{\partial x} = \sum_{k \neq i,j} \varepsilon_{ikj} Q_{ik} Q_{kj} \quad (10.22)$$

$$\varepsilon_{ikj} = a_i b_k - a_k b_i + a_j b_i - a_i b_j + a_k b_j - a_j b_k \quad (10.23)$$

Let I – some real diagonal matrix $I^2 = 1$. All diagonal elements $I_{kk} = \pm 1$. Suppose that Q satisfies the condition

$$Q^+ = IQI \quad (10.24)$$

This condition is called reduction.

If $n = 3$, completely anti-symmetric tensor ε_{ikj} has only one component $\varepsilon = \varepsilon_{123}$. Let us assume that $I = 1, a_1 > a_2 > a_3$

$$\begin{aligned} Q_{13} &= \frac{i}{\sqrt{a_1 - a_3}} A \\ Q_{12} &= \frac{i}{\sqrt{a_1 - a_2}} B \\ Q_{23} &= \frac{i}{\sqrt{a_2 - a_3}} C \\ Q_{ij} &= -i Q_{ji}^* \end{aligned} \quad (10.25)$$

We go to the three wave system

$$\begin{aligned}
 V_{ij}^{(1)} &= \frac{a_j b_i - a_i b_j}{a_i - a_j} \\
 V_{ij}^{(2)} &= \frac{b_i - b_j}{a_i - a_j} \\
 u &= \frac{\varepsilon_{123}}{\sqrt{(a_1 - a_2)(a_1 - a_3)(a_2 - a_3)}}
 \end{aligned}$$

Assuming that

$$\begin{aligned}
 I &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 Q_{13} &= \frac{iA^*}{\sqrt{a_1 - a_3}} \\
 Q_{12} &= \frac{iB}{\sqrt{a_1 - a_2}} \\
 Q_{13} &= \frac{iC}{\sqrt{a_2 - a_3}}
 \end{aligned}$$

We obtain the explosive system (9.33).

Lecture 11

Dressing method

Let $w(z) = u(x, y) + iv(x, y)$ ($z = x + iy$) be an analytic function. Then

$$\frac{\partial w}{\partial \bar{z}} = \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) w = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = 0$$

in virtue of Cauchy-Riemann conditions. What is

$$\frac{\partial}{\partial \bar{z}} \frac{1}{z} ?$$

To answer this question we present

$$\frac{\partial}{\partial \bar{z}} \frac{\bar{z}}{z\bar{z} + \varepsilon} = \frac{1}{z\bar{z} + \varepsilon} - \frac{z\bar{z}}{(z\bar{z} + \varepsilon)^2} = \frac{\varepsilon}{(z\bar{z} + \varepsilon)^2} \quad (11.1)$$

If $\varepsilon \rightarrow 0$, expression (11.1) tends to zero everywhere except $z = 0$. One could guess that it is a δ -function of two variables with some coefficient. We introduce polar coordinates now:

$$\frac{\partial}{\partial \bar{z}} \frac{1}{z} = \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{(r^2 + \varepsilon)^2}$$

Integrating over the whole complex plane one gets after integrating over angles

$$2\pi \int_0^\infty \frac{\varepsilon}{(r^2 + \varepsilon)^2} r dr = \pi \varepsilon \int_0^\infty \frac{du}{(u + \varepsilon)^2} = \pi$$

Finally

$$\frac{\partial}{\partial \bar{z}} \frac{1}{z} = \pi \delta(z) \delta(\bar{z}) = \pi \delta(x) \delta(y) \quad (11.2)$$

In the same way for $z_0 = x_0 + iy_0$

$$\frac{\partial}{\partial \bar{z}} \frac{1}{z - z_0} = \pi \delta(x - x_0) \delta(y - y_0) \quad (11.3)$$

Thereafter we will denote complex numbers by Greek letters.

Suppose that function $\chi(\lambda, \bar{\lambda})$ is not analytic but satisfies on the λ -plane the following condition (nonlocal $\bar{\partial}$ -problem)

$$\frac{\partial \chi}{\partial \bar{\lambda}} = \int \chi(\eta, \bar{\eta}) T(\eta, \bar{\eta}, \lambda, \bar{\lambda}) d\eta d\bar{\eta} \quad (11.4)$$

at $\lambda \rightarrow \infty$ this function tends to the constant $\chi \rightarrow \chi_0$.

Using formula (11.3) one can transform (11.4) to integral equation

$$\chi = \chi_0 + \frac{1}{\pi} \int \frac{T(\xi, \bar{\xi}, \eta, \bar{\eta})}{\lambda - \xi} \chi(\eta, \bar{\eta}) d\eta d\bar{\eta} d\xi d\bar{\xi} \quad (11.5)$$

Singularity

$$\frac{1}{\lambda - \xi} = \lim_{\varepsilon \rightarrow 0} \frac{\bar{\lambda} - \bar{\xi}}{|\lambda - \xi|^2 + \varepsilon}.$$

According to Fredholm alternative, equation (11.5) has nontrivial unique solution if and only if homogeneous equation

$$\chi = \frac{1}{\pi} \int \frac{T(\xi, \bar{\xi}, \eta, \bar{\eta})}{\lambda - \xi} \chi(\eta, \bar{\eta}) d\eta d\bar{\eta} \quad (11.6)$$

has only trivial zero solution. We will assume that this condition is satisfied. It means that the nonlocal $\bar{\partial}$ -problem (11.4) normalized by zero condition in infinity

$$\chi \rightarrow 0 \text{ as } \lambda \rightarrow \infty$$

has only zero solution.

We assume now that the kernel T in (11.4) depends on coordinates x, y, t as follow

$$\begin{aligned} T(\eta, \bar{\eta}, \lambda, \bar{\lambda}) &= e^{\Phi(\eta)} T_0(\eta, \bar{\eta}, \lambda, \bar{\lambda}) e^{-\Phi(\lambda)} \\ \Phi(\lambda) &= i\lambda x + \lambda^2 y + 4i\lambda^3 t \end{aligned} \quad (11.7)$$

It means that T satisfies the system of linear equations

$$\begin{aligned}\frac{\partial T}{\partial x} + i(\lambda - \eta)T &= 0 \\ \frac{\partial T}{\partial y} + i(\lambda^2 - \eta^2)T &= 0 \\ \frac{\partial T}{\partial t} + i(\lambda^3 - \eta^3)T &= 0\end{aligned}\tag{11.8}$$

Equation (11.4) can be rewritten symbolically as follows

$$\frac{\partial \chi}{\partial \lambda} = \chi * T\tag{11.9}$$

Let us introduce following differential operations

$$\begin{aligned}D_1\chi &= \frac{\partial \chi}{\partial x} + i\lambda\chi \\ D_2\chi &= \frac{\partial \chi}{\partial y} + \lambda^2\chi\end{aligned}\tag{11.10}$$

$$D_3\chi = \frac{\partial \chi}{\partial t} + 4i\lambda^3\chi\tag{11.11}$$

They commute with operation $\partial/\partial\bar{\lambda}$. Applying D_i to (11.9) one can see that

$$\frac{\partial}{\partial \bar{\lambda}} D_i \chi = D_i \chi * T\tag{11.12}$$

Let $P\chi = P(D_1, D_2, D_3)\chi$ is any polynomial on D_1, D_2, D_3 . Its coefficients are functions on (λ, y, t) . By induction one can realize that

$$\frac{\partial}{\partial \bar{\lambda}} P\chi = P\chi * T\tag{11.13}$$

Let $\chi_0 = 1$, $\lambda \rightarrow \infty$. In neighborhood of infinity function χ has asymptotic expansion

$$\chi = 1 + \frac{\chi_1}{\lambda} + \frac{\chi_2}{\lambda^2} + \dots\tag{11.14}$$

$$\begin{aligned}\chi_1 &= \frac{1}{\pi} \int T(\xi, \bar{\xi}, \eta, \bar{\eta}) \chi(\eta, \bar{\eta}) d\eta d\bar{\eta} d\xi d\bar{\xi} \\ \chi_2 &= \frac{1}{\pi} \int \xi T(\xi, \bar{\xi}, \eta, \bar{\eta}) \chi(\eta, \bar{\eta}) d\eta d\bar{\eta} d\xi d\bar{\xi}\end{aligned}\tag{11.15}$$

and so on.

Now we construct following differential operator

$$\begin{aligned}
P_1\chi &= (D_2 + D_1^2 + u) \chi \\
P_1\chi &= \frac{\partial\chi}{\partial y} + \lambda^2\chi + \left(\frac{\partial}{\partial x} + i\lambda\right)^2 \chi + u\chi = \\
&= \frac{\partial\chi}{\partial y} + \frac{\partial^2\chi}{\partial x^2} + 2i\lambda\frac{\partial\chi}{\partial x} + u\chi
\end{aligned} \tag{11.16}$$

substituting (11.14) in (11.16) we see that at $\lambda \rightarrow \infty$

$$P_1\chi \Rightarrow 2i\frac{\partial\chi_1}{\partial x} + u + o\left(\frac{1}{\lambda}\right)$$

Hence if

$$\begin{aligned}
u &= -2i\frac{\partial\chi_1}{\partial x} \\
P_1\chi &\rightarrow 0 \quad \text{at } \lambda \rightarrow \infty
\end{aligned} \tag{11.17}$$

However, $P_1\chi$ is a solution of the nonlocal $\bar{\partial}$ -problem (11.13). Hence $P_1\chi = 0$ and function χ satisfies the equation

$$(D_2 + D_1^2 + u) \chi = 0 \tag{11.18}$$

One should remember that u is defined by equation (11.17)

In the same way we construct operator

$$\begin{aligned}
P_2\chi &= (D_3 + 4D_1^3 + 6VD_1 + 3V_x + q) \chi = \\
&= \left(\frac{\partial}{\partial t} - 12\lambda^2\frac{\partial}{\partial x} + 12i\lambda\frac{\partial^2}{\partial x^2} + 4\frac{\partial^3}{\partial x^3} + \right. \\
&\quad \left. + 6V\left(\frac{\partial}{\partial x} + i\lambda\right) + (3V_x + q)\right) \chi
\end{aligned} \tag{11.19}$$

Substituting (11.14) into (11.19) and send $\lambda \rightarrow \infty$ one can see that

$$P_2\chi \rightarrow \lambda \left[\left(-12\frac{\partial\chi_1}{\partial x}\right) + 6iV \right] + 12i\frac{\partial^2\chi_1}{\partial x^2} + 3V_x + q - 12\frac{\partial\chi_2}{\partial x}$$

If we choose

$$\begin{aligned} V = u &= -2i \frac{\partial \chi_1}{\partial x} \\ q &= 12 \frac{\partial \chi_2}{\partial x} - 6i \frac{\partial^2 \chi_1}{\partial x^2} \end{aligned} \quad (11.20)$$

We achieve $P_2 \chi \rightarrow 0$ as $\lambda \rightarrow \infty$. Hence $P_2 \chi = 0$ or

$$(D_3 + 4D_1^3 + 6uD_1 + 3u_x + q) \chi = 0 \quad (11.21)$$

Let

$$\begin{aligned} \chi &= \varphi e^{-i\lambda x - \lambda^2 y - 4i\lambda^3 t} \\ D_1 \chi &= \frac{\partial \varphi}{\partial x} e^{-i\lambda x - \lambda^2 y - 4i\lambda^3 t} \\ D_2 \chi &= \frac{\partial \varphi}{\partial y} e^{-i\lambda x - \lambda^2 y - 4i\lambda^3 t} \\ D_3 \chi &= \frac{\partial \varphi}{\partial t} e^{-i\lambda x - \lambda^2 y - 4i\lambda^3 t} \end{aligned}$$

In terms of φ equations (11.18), (11.21) read

$$\begin{aligned} \frac{\partial \varphi}{\partial y} + \frac{\partial^2 \varphi}{\partial x^2} + u\varphi &= 0 \\ \frac{\partial \varphi}{\partial t} + 4 \frac{\partial^3 \varphi}{\partial x^3} + 6u \frac{\partial \varphi}{\partial x} + (3u_x + q)\varphi &= 0 \end{aligned} \quad (11.22)$$

This is the Lax pair for KP-2 equation. Hence u satisfies the KP-2 equation.

If we choose

$$\begin{aligned} D_1 &= \frac{\partial}{\partial x} + i\lambda \\ D_2 &= \frac{\partial}{\partial y} + i\lambda^2 \\ D_3 &= \frac{\partial}{\partial t} + 4i\lambda^3 \end{aligned}$$

We end up with KP-1 equation. Now $\Phi(\lambda) = i(\lambda x + \lambda^2 y + 4\lambda^3 t)$.

In the linear approximation one can put $\chi = 1$ in (11.14) (11.15). Then

$$\chi_1 \simeq \frac{1}{\pi} \int T_0(\xi, \bar{\xi}, \eta, \bar{\eta}) e^{i(\xi - \eta)x + i(\xi^2 - \eta^2)y + 4i(\xi^3 - \eta^3)t} d\xi d\bar{\xi} d\eta d\bar{\eta} \quad (11.23)$$

Now we see the origin of the parametrization (10.5)

Lecture 12

Solitonic solutions of the KdV equation

Suppose that the kernel in $\bar{\partial}$ -problem is such that y -dependence drops out. To do this, one can put

$$T(\eta, \bar{\eta}, \lambda, \bar{\lambda}) = T(\lambda, \bar{\lambda})\delta(\eta + \lambda)\delta(\bar{\eta} + \bar{\lambda}) \quad (12.1)$$

Function χ satisfies now the following $\bar{\partial}$ -problem

$$\frac{\partial \chi}{\partial \bar{\lambda}} = T(\lambda, \bar{\lambda})e^{2i(\lambda x + 4\lambda^3 t)}\chi(-\lambda, -\bar{\lambda}) \quad (12.2)$$

So far $T(\lambda, \bar{\lambda})$ -is an arbitrary analytic complex function of λ .

This equation can be solved explicitly if one assumes:

$$T(\lambda, \bar{\lambda}) = \pi \sum_{n=1}^N \delta(\lambda - \lambda_n)\delta(\bar{\lambda} - \bar{\lambda}_n) \quad (12.3)$$

here $\lambda_1, \dots, \lambda_n$ -set of complex numbers.

From (12.2) one can see that χ is analytic function in all complex plane with exception the set of points $\lambda_1, \dots, \lambda_n$. In this points it has simple poles. Thus χ is a rational function

$$\chi = 1 + \sum_{m=1}^N \frac{f_m}{\lambda - \lambda_m} \quad (12.4)$$

$$f_n = f_n(x, t)$$

Let us denote $\phi_n = (\lambda_n x + 4\lambda_n^3 t) = \phi_n(\lambda, t)$ Using the Poincaré formula one gets

$$f_n = T_n e^{2i\phi_n} \chi|_{-\lambda_n} \quad (12.5)$$

$$\chi|_{-\lambda_n} = 1 - \sum_{m=1}^N \frac{f_m}{\lambda_n + \lambda_m} \quad (12.6)$$

Now we have to assume that $\lambda_n + \lambda_m \neq 0$. In other words the set of poles does not include reflected points $\lambda_n \rightarrow -\lambda_n$. Equation (12.5) reads

$$f_n + T_n e^{2i\phi_n} \sum_{m=1}^N \frac{f_m}{\lambda_n + \lambda_m} = T_n e^{2i\phi_n} \quad (12.7)$$

Function χ (12.4) is the following expansion at $\lambda \rightarrow \infty$

$$\chi = 1 + \frac{1}{\lambda} \sum f_m + \frac{1}{\lambda^2} \sum \lambda_m f_m + \dots$$

Hence $\chi_1 = \sum_{m=1}^N f_m$, $\chi_2 = \sum_{m=1}^N \lambda_m f_m$, etc.

Suppose that $N = 1$, $\lambda_1 = \mu$

$$\chi = 1 + \frac{f}{\lambda - \mu}, \quad \phi_1 = \phi = (\mu x + 4\mu^3 t), \quad T_1 = T, \quad f_1 = f.$$

$$f = \frac{T}{e^{-2i\phi} + \frac{T}{2\mu}}$$

$$u = -2i \frac{\partial f}{\partial x} = \frac{4\mu T e^{-2i\phi}}{(e^{-2i\phi} + \frac{T}{2\mu})^2} \quad (12.8)$$

This is one-solitonic solution of the complex KdV equation. It depends on two complex parameters μ , T . To make it a regular solution of the real KdV equation, one has to put

$$\mu = i\kappa \quad \kappa \text{ is real}$$

$$T = i\kappa e^{2\kappa x_0} \quad x_0 \text{ is real .}$$

Then solution (12.8) reads $u = -\frac{2\kappa^2}{\cosh^2[\kappa(x-x_0)]}$.

This is a soliton!

Lecture 13

N -soliton solutions of the KdV equation

To construct N -solitonic solutions, one has to assume that $\lambda_k = i\kappa_k$, $\kappa_k > 0$ and

$$T_k = iM_k^2 \quad f_k = ig_k \quad i\phi_k = L_k = -(\kappa_k x + 4\kappa_k^3 t)$$

Equation (12.7) reads

$$\begin{aligned} g_k + M_k^2 e^{2L_k} \sum_{m=1}^N \frac{g_m}{\kappa_k + \kappa_m} &= M_k^2 e^{2L_k} \\ \chi_1 &= i \sum_{m=1}^N g_m, \quad g_k = h_k e^{L_k} \\ h_k + M_k^2 \sum_{m=1}^N \frac{h_m e^{L_k + L_m}}{\kappa_k + \kappa_m} &= M_k^2 e^{L_k} \end{aligned} \tag{13.1}$$

Absolutely central role in future consideration plays Theorem 1.

Theorem 1

$$\chi_1 = -i \frac{\partial}{\partial x} \ln \Delta \tag{13.2}$$

$$\Delta = \det \left\| \delta_{km} + \frac{M_k^2 e^{L_k + L_m}}{\kappa_k + \kappa_m} \right\| \tag{13.3}$$

In the world literature Δ is called τ -function! First we prove Theorem 1. One can see that $\tau = \Delta$ (13.3) is just a determinate of system (13.1)

$$\frac{\partial}{\partial x} \ln \tau = \frac{\tau_x}{\tau}$$

To find a derivative from a determinate one should differentiate sequentially columns of the determinate and add the results. Then

$$\frac{\tau_x}{\tau} = \sum_m \frac{\tau_m}{\tau}$$

τ_m – result of differentiation of the column number m . In τ_m only this column differs from columns in τ .

$$\tau_m = \begin{pmatrix} -M_1^2 e^{L_1} \\ -M_2^2 e^{L_2} \\ \vdots \\ -M_n^2 e^{L_n} \end{pmatrix} e^{L_m} = -h_m e^{L_m}$$

here we used Cramer's theorem.

Finally

$$\frac{\tau_x}{\tau} = - \sum h_m e^{L_m}$$

Comparing with (13.2) we accomplish the proof. Note that

$$u = -2 \frac{\partial^2}{\partial x^2} \ln \tau = -2 \frac{\tau_x^2 - \tau \tau_{xx}}{\tau^2} \quad (13.4)$$

Let us look at the structure of τ -function. One can see that it can be presented in the following form

$$\tau = 1 + \tau_1 + \tau_2 + \dots \quad (13.5)$$

Here

$$\tau_1 = \sum \frac{M_k^2}{2\kappa_k}$$

$$\tau_2 = \sum_{i < j} M_i^2 M_j^2 \Delta_{ij}^{(2)}$$

$$\Delta_{ij} = \left| \begin{array}{cc} \frac{1}{2\kappa_i} & \frac{1}{\kappa_i + \kappa_j} \\ \frac{1}{\kappa_i + \kappa_j} & \frac{1}{2\kappa_j} \end{array} \right| = \frac{1}{4\kappa_i \kappa_j} - \frac{1}{(\kappa_i + \kappa_j)^2} = \frac{(\kappa_i - \kappa_j)^2}{4\kappa_i \kappa_j (\kappa_i + \kappa_j)^2}$$

$$\tau_3 = \sum_{i < j < k} M_i^2 M_j^2 M_k^2 \Delta_{ijk}$$

$$\Delta_{ijk} = \left| \begin{array}{ccc} \frac{1}{2\kappa_i} & \frac{1}{\kappa_i + \kappa_j} & \frac{1}{\kappa_i + \kappa_k} \\ \frac{1}{\kappa_i + \kappa_j} & \frac{1}{2\kappa_j} & \frac{1}{\kappa_j + \kappa_k} \\ \frac{1}{\kappa_i + \kappa_k} & \frac{1}{\kappa_j + \kappa_k} & \frac{1}{2\kappa_k} \end{array} \right|$$

To calculate Δ_{ijk} we note first that this expression is symmetric with respect to permutation of its indices. Indeed

$$\Delta_{ijk} = \left| \begin{array}{ccc} \frac{1}{2\kappa_i} & \frac{1}{\kappa_i + \kappa_j} & \frac{1}{\kappa_i + \kappa_k} \\ \frac{1}{\kappa_i + \kappa_j} & \frac{1}{2\kappa_j} & \frac{1}{\kappa_j + \kappa_k} \\ \frac{1}{\kappa_i + \kappa_k} & \frac{1}{\kappa_j + \kappa_k} & \frac{1}{2\kappa_k} \end{array} \right| = \left| \begin{array}{ccc} \frac{1}{\kappa_i + \kappa_j} & \frac{1}{2\kappa_j} & \frac{1}{\kappa_j + \kappa_k} \\ \frac{1}{2\kappa_i} & \frac{1}{\kappa_i + \kappa_j} & \frac{1}{\kappa_i + \kappa_k} \\ \frac{1}{\kappa_i + \kappa_k} & \frac{1}{\kappa_j + \kappa_k} & \frac{1}{2\kappa_k} \end{array} \right|$$

In the last formula we replace first and second rows. Let us replace first and second columns.

$$\Delta_{ijk} = \left| \begin{array}{ccc} \frac{1}{2\kappa_j} & \frac{1}{\kappa_i + \kappa_j} & \frac{1}{\kappa_j + \kappa_k} \\ \frac{1}{\kappa_i + \kappa_j} & \frac{1}{2\kappa_i} & \frac{1}{\kappa_i + \kappa_k} \\ \frac{1}{\kappa_j + \kappa_k} & \frac{1}{\kappa_i + \kappa_k} & \frac{1}{2\kappa_k} \end{array} \right| = \Delta_{jik}$$

Then Δ_{ijk} can be presented in the form

$$\Delta_{ijk} = \frac{P_{ijk}}{8\kappa_i \kappa_j \kappa_k (\kappa_i + \kappa_k)^2 (\kappa_i + \kappa_j)^2 (\kappa_j + \kappa_k)^2}$$

Here P_{ijk} -symmetric polynomial of power 6. Apparently $\Delta_{iik} = 0$ Hence $P_{ijk} \simeq \lambda(\kappa_i - \kappa_j)^2 (\kappa_i - \kappa_k)^2 (\kappa_j - \kappa_k)^2$, where λ - is still indefinite constant. To find it, one set $\kappa_i \rightarrow 0$. In this limit

$$\Delta_{ijk} \simeq \frac{1}{2\kappa_i} \Delta_{jk}$$

Hence $\lambda = 1$. Finally

$$\Delta_{ijk} = \frac{(\kappa_i - \kappa_j)^2 (\kappa_i - \kappa_k)^2 (\kappa_j - \kappa_k)^2}{8\kappa_i \kappa_j \kappa_k (\kappa_i + \kappa_k)^2 (\kappa_i + \kappa_j)^2 (\kappa_j + \kappa_k)^2}$$

Let us introduce

$$\frac{M_i^2}{2\kappa_i} = e^{2\kappa_i x_i} \quad x_i = \frac{1}{2\kappa_i} \ln \frac{M_i^2}{2\kappa_i}$$

Now

$$\begin{aligned} \tau_1 &= \sum_k e^{2\phi_k} \quad \phi_k = -\kappa_k(x - x_k) + 4\kappa_k^3 t \\ \tau_2 &= \sum_{i < j} e^{2(\phi_i + \phi_j)} \frac{(\kappa_i - \kappa_j)^2}{(\kappa_i + \kappa_j)^2} \end{aligned}$$

In the same way

$$\tau_l = \sum_{i_1 < i_2 < \dots < i_l} e^{2(\phi_{i_1} + \phi_{i_2} + \dots + \phi_{i_l})} \prod_{m > k} \frac{(\kappa_{i_k} - \kappa_{i_m})^2}{(\kappa_{i_k} + \kappa_{i_m})^2} \quad (13.6)$$

where $k = 1 \dots l$ and $m = 1 \dots l$

Remark

Note that we can remove the restriction $\kappa_i > 0$. What we need is $\frac{M_i^2}{2\kappa_i} > 0$ and $\kappa_i + \kappa_j \neq 0$

If we have exactly n poles

$$\begin{aligned} \tau &= 1 + \tau_1 + \dots + \tau_n \\ \tau_n &= e^{2(\phi_1 + \dots + \phi_n)} \prod \frac{(\kappa_i - \kappa_j)^2}{(\kappa_i + \kappa_j)^2}, \quad j > i \end{aligned} \quad (13.7)$$

where $i = 1 \dots n$ and $j = 1 \dots n$. One can see that two-solitonic solution is defined by four parameters $\kappa_1, \kappa_2, x_1, x_2$

$$\tau = 1 + e^{2\phi_1} + e^{2\phi_2} + \frac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2} e^{2(\phi_1 + \phi_2)} \quad (13.8)$$

Here

$$\begin{cases} \phi_1 = -\kappa_1[(x - x_1 - 4\kappa_1^2 t)] \\ \phi_2 = -\kappa_2[(x - x_2 - 4\kappa_2^2 t)] \end{cases}$$

In virtue of (13.4) following transform

$$\tau \rightarrow a e^{bx} \tau, \quad a, b - \text{constants}$$

does not change u .

Let in (13.8) $\phi_1 \rightarrow -\infty$, ϕ_2 – finite. Then $\tau \rightarrow 1 + e^{2\phi_2}$. This is solution with parameters κ_2, x_2 . If $\phi_1 \rightarrow e^{2\phi_1} \left(1 + \frac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2} e^{2\phi_2}\right)$. Factor $e^{2\phi_1}$ can be omitted, and one can put

$$\begin{aligned}\tau &\rightarrow 1 + \frac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2} e^{2\phi_2} = 1 + e^{2\tilde{\phi}_2} \\ \tilde{\phi}_2 &= -\kappa_2(x - \tilde{x}_2) + 4\kappa_2^3 t \\ \tilde{x}_2 &= x_2 - \frac{1}{2\kappa_2} \ln \frac{(\kappa_1 + \kappa_2)^2}{(\kappa_1 - \kappa_2)^2}\end{aligned}$$

In the same way

if $\phi_2 \rightarrow -\infty$ ϕ_1 – finite

$\tau \rightarrow 1 + e^{2\phi_1}$ parameters κ_1, x_1

if $\phi_2 \rightarrow \infty$

$\tau \rightarrow 1 - \frac{(\kappa_1 + \kappa_2)^2}{(\kappa_1 - \kappa_2)^2} e^{2\phi_1}$ parameters $\kappa_1, x_1 + \frac{1}{2\kappa_1} \ln \frac{(\kappa_1 + \kappa_2)^2}{(\kappa_1 - \kappa_2)^2}$ Now let $\kappa_1 > \kappa_2 > 0$, $t \rightarrow \pm\infty$.

If ϕ_1 is finite $x \simeq 4\kappa_1^2 t$

$$\begin{aligned}\phi_2 &\simeq -4\kappa_2(\kappa_1^2 - \kappa_2^2)t \\ \phi_2 &\rightarrow -\infty \text{ if } t \rightarrow +\infty \\ \phi_2 &\rightarrow +\infty \text{ if } t \rightarrow -\infty\end{aligned}$$

If ϕ_2 is finite $x \simeq 4\kappa_2^2 t$

$$\phi_1 \simeq -4\kappa_1(\kappa_2^2 - \kappa_1^2)t = 4\kappa_2(\kappa_1^2 - \kappa_2^2)t$$

Now situation is opposite

$$\begin{aligned}\phi_1 &\rightarrow +\infty \text{ if } t \rightarrow -\infty \\ \phi_1 &\rightarrow -\infty \text{ if } t \rightarrow +\infty\end{aligned}$$

Summarizing the situation we see that if $t \rightarrow -\infty$ the "fast" solution is posed at

$$x = 4\kappa_1^2 t - x_1 - \frac{1}{2\kappa_1} \ln \frac{(\kappa_1 + \kappa_2)^2}{(\kappa_1 - \kappa_2)^2} \quad (13.9)$$

the slow solution is posed at

$$x = 4\kappa_2^2 t - x_2$$

If $t \rightarrow +\infty$ the fast solution is posed at

$$x \simeq 4\kappa_1^2 t - x_1$$

The slow solution is posed at

$$x \simeq 4\kappa_2^2 t - x_2 - \frac{1}{2\kappa_2} \ln \frac{(\kappa_1 + \kappa_2)^2}{(\kappa_1 - \kappa_2)^2} \quad (13.10)$$

These results can be interpreted as follow. Solutions interact like repelling particles. They scatter elastically. "Fast" solution chases "slow" one and hits it. Then "slow" solution turns to "fast" one.

In the same way one can study time asymptotics of the N-soliton solution. Let $\kappa_1 > \kappa_2 > \dots > \kappa_n > 0$. We will study asymptotic behavior of the soliton with number k .

$$\begin{aligned} \phi_k &\simeq \text{const} & x &\simeq 4\kappa_k^2 t \\ \phi_m &= -4\kappa_m(\kappa_k^2 - \kappa_m^2)t \\ &\text{If } t \rightarrow -\infty \\ \phi_m &\rightarrow -\infty \text{ if } m > k \\ \phi_m &\rightarrow +\infty \text{ if } m < k \end{aligned}$$

The most important are two terms, one is proportional to $e^{2(\phi_1 + \dots + \phi_{k-1})}$, second one is proportional to $e^{2(\phi_1 + \dots + \phi_{k-1} + \phi_k)}$ Keeping in consideration only these two terms and making "cancelling" of an insignificant factor ae^x one gets

$$\tau \simeq 1 + \frac{\prod(\kappa_1 - \kappa_k)^2 \dots (\kappa_{k-1} - \kappa_k)^2}{\prod(\kappa_1 + \kappa_k)^2 \dots (\kappa_{k+1} + \kappa_k)^2} e^{2\phi_k}$$

It means that at $t \rightarrow -\infty$ the "k-soliton" is posed at

$$x_k^- \simeq 4\kappa_k^2 t - x_k + \sum_{l=1}^{k-1} \frac{1}{2\kappa_k} \ln \frac{(\kappa_l + \kappa_k)^2}{(\kappa_l - \kappa_k)^2} \quad (13.11)$$

If $t \rightarrow +\infty$

$$\begin{aligned} \phi_m &\rightarrow -\infty, \text{ if } m < k, \\ \phi_m &\rightarrow +\infty, \text{ if } m > k. \end{aligned}$$

Repeating previous consideration one can find that the "k-soliton" is posed

$$x_k^+ \simeq 4\kappa_k^2 t - x_k + \sum_{l=k+1}^n \frac{1}{2\kappa_k} \ln \frac{(\kappa_l + \kappa_k)^2}{(\kappa_l - \kappa_k)^2} \quad (13.12)$$

The total shift for the "k-soliton" is

$$x_k^+ - x_k^- = \frac{1}{2\kappa_k} \sum_{l=1}^{k-1} \ln \frac{(\kappa_l + \kappa_k)^2}{(\kappa_l - \kappa_k)^2} - \sum_{l=k+1}^n \ln \frac{(\kappa_l + \kappa_k)^2}{(\kappa_l - \kappa_k)^2}$$

We obtained a remarkable result. A soliton of an indemediated velocity acquired positive shift after interaction with slower solitons and negative shift after interaction with faster solitons. A total shift is algebraic sum of particle shifts. This shift does not depend on details of interaction.

If two solitons have very close parameters κ_1, κ_2 they could not get so close to each other. The minimal distance between them is proportional to

$$\Delta x \simeq \frac{1}{2\kappa_1} \ln \frac{(\kappa_1 - \kappa_2)^2}{4\kappa_1^2}$$

Lecture 14

Unchecked. Scattering in the Schrödinger equation

We start with equation:

$$\frac{d^2}{dx^2}\Psi + k^2\Psi = u(x)\Psi \quad -\infty < k < \infty \quad (14.1)$$

$u(x)$ -real function satisfying the condition

$$\int_{-\infty}^{\infty} (1 + |x|)|u(x)|dx < \infty \quad (14.2)$$

$k = k_n$ is eigenvalue if the solution f_n of equation (14.1) tends to zero at $|x| \rightarrow \infty$. It is well known that this solution is unique. Indeed, if Ψ_1, Ψ_2 are two solutions of (14.1) then

$$\{\Psi_1, \Psi_2\} = \text{const} = C \quad (14.3)$$

Here $\{\Psi_1, \Psi_2\} = \Psi_{1x}\Psi_2 - \text{Psi}_{2x}\Psi_1$ -wronskian of functions Ψ_1, Ψ_2 . If Ψ_1, Ψ_2 -eigenfunctions, they tend to zero at $|x| \rightarrow \infty$, hence $C = 0$ and Ψ_1, Ψ_2 are proportional to each other.

Eigenvalue k_n must be pure imaginary. Indeed, if k_n is complex

$$\begin{aligned}\frac{d^2 f_n}{dx^2} + k_n^2 f_n &= u f \\ \frac{d^2 \bar{f}_n}{dx^2} + k_n^2 \bar{f}_n &= u \bar{f}\end{aligned}\tag{14.4}$$

From (14.4) one gets

$$\frac{d}{dx}\{f_n, \bar{f}_n\} = (\bar{k}_n^2 - k_n^2)|f_n|^2\tag{14.5}$$

after integrating by x one obtains

$$\bar{k}_n^2 = k_n^2$$

Apparently $\{f_n, \bar{f}_n\} = 0$, and eigenfunction F can be made real. Let us introduce Jost functions Ψ, Φ -solutions of equation (14.1), defined by boundary conditions

$$\begin{aligned}\Psi &\rightarrow e^{ikx} & \Phi &\rightarrow e^{-ikx} \\ x &\rightarrow +\infty & x &\rightarrow -\infty\end{aligned}\tag{14.6}$$

Jost functions satisfy certain integral equations. One can present Ψ in a form

$$\Psi = c_1 e^{ikx} + c_2 e^{-ikx}$$

with additional condition

$$c_1' e^{ikx} + c_2' e^{-ikx} = 0\tag{14.7}$$

Hence

$$\begin{aligned}\Psi' &= ik(c_1 e^{ikx} - c_2 e^{-ikx}) \\ \Psi'' + k^2 \Psi &= ik(c_1' e^{ikx} - c_2' e^{-ikx}) = u \Psi\end{aligned}\tag{14.8}$$

Combining (14.7), (14.8), one gets

$$c_1' = \frac{1}{2ik} u \Psi e^{-ikx} \quad c_2' = -\frac{1}{2ik} u \Psi e^{ikx}\tag{14.9}$$

Integrating equation (14.9) we take into account boundary conditions

$$\begin{aligned}
c_1 &= 1 - \frac{1}{2ik} \int_x^\infty u \Psi e^{-iky} dy \\
c_2 &= \frac{1}{2ik} \int_x^\infty u \Psi e^{iky} dy
\end{aligned} \tag{14.10}$$

One can introduce a new function $A = \Psi e^{-ikx} = c_1 + c_2 e^{-2ikx}$. From (14.10) we conclude that A satisfies the integral equation

$$A(x, k) = 1 - \frac{1}{2ik} \int_x^\infty u(y) (1 - e^{2ik(y-x)}) A(k, y) dy \tag{14.11}$$

The same operation can be performed with function Φ . Now

$$\begin{aligned}
c_1 &= \frac{1}{2ik} \int_x^\infty u \Phi e^{-iky} dy \\
c_2 &= 1 - \frac{1}{2ik} \int_{-\infty}^x u \Phi e^{iky} dy.
\end{aligned} \tag{14.12}$$

Let us denote $B = \Phi e^{ikx}$. This function satisfies the integral equation

$$B(x, k) = 1 - \frac{1}{2ik} \int_x^\infty u (1 - e^{2ik(x-y)}) B(k, y) dy \tag{14.13}$$

Suppose now that $k = \xi + i\eta$, $\eta > 0$

$$|e^{2ik(y-x)}| = e^{-2\eta(y-x)}$$

In (14.2) $y > x$ and this exponent tends to zero as $y \rightarrow \infty$. In (14.13) $|e^{2ik(x-y)}| = e^{-2\eta(x-y)}$. As far as $y < x$, this exponent also tends to zero $\eta \rightarrow \infty$.

Hence both functions A, B could be analytically continued to the upper-plane. They have these asymptotic expansions

$$\begin{aligned}
A &\rightarrow 1 - \frac{1}{2ik} \int_x^\infty u(y) dy \\
B &\rightarrow 1 - \frac{1}{2ik} \int_{-\infty}^x u(y) dy
\end{aligned} \quad k \rightarrow \infty \quad \text{Im} k > 0 \tag{14.14}$$

and

$$\Psi \rightarrow e^{ikx} \left(1 - \frac{1}{2ik} \int_x^\infty u(y) dy \right) \quad \Phi \rightarrow e^{-ikx} \left(1 - \frac{1}{2ik} \int_{-\infty}^x u(y) dy \right)$$

Let $k = i\kappa_n$. Then

$$\begin{aligned} \Psi|_{k=i\kappa_n} &\rightarrow e^{-\kappa_n x} & x &\rightarrow \infty \\ \Phi|_{k=i\kappa_n} &\rightarrow e^{\kappa_n x} & x &\rightarrow -\infty \end{aligned}$$

They present the same eigenfunction f_n and can differ only on some factor.

Suppose that f_n is designed by asymptotic

$$\begin{aligned} f_n &\rightarrow e^{\kappa_n x} & x &\rightarrow -\infty \\ f_n &\rightarrow b_n e^{\kappa_n x} & x &\rightarrow \infty \end{aligned} \quad (14.15)$$

Hence

$$f_n = \Phi|_{k=i\kappa_n} = b_n \Psi|_{k=i\kappa_n}. \quad (14.16)$$

In this point Ψ and Φ are proportional to each other.

$\bar{\Psi}(k, x) = \Psi(-k, x)$ and $\bar{\Phi}(k, x) = \Phi(-k, x)$ also are solutions of equation (14.1). Apparently, they are analytic in lower half-plane. Solutions $\Psi, \bar{\Psi}$ comprise a fundamental system. Then, one can put

$$\begin{aligned} \Phi(k, x) &= a(k)\Psi(-k, x) + b(k)\Psi(k, x) \\ \Phi(-k, x) &= b(-k)\Psi(-k, x) + a(-k)\Psi(k, x) \end{aligned} \quad (14.17)$$

Apparently

$$a(-k) = \bar{a}(k) \quad b(-k) = \bar{b}(k) \quad (14.18)$$

Note that

$$\{\Psi(k), \Psi(-k)\} = 2ik \quad \{\Phi(k), \Phi(-k)\} = -2ik \quad (14.19)$$

Calculating $\{\Phi(k), \Phi(-k)\}$ by the use of (14.17) one finds

$$|a(k)|^2 - |b(k)|^2 = 1. \quad (14.20)$$

We will call $\begin{vmatrix} a & b \\ \bar{b} & \bar{a} \end{vmatrix}$ -monodromy matrix, according to (14.20) this matrix is unimodular.

Now from (14.17), (14.19) we get

$$a(k) = \frac{1}{2ik} \{\Psi, \Phi\} \quad \bar{a}(k) = \frac{1}{2ik} \{\bar{\Psi}, \bar{\Phi}\} \quad (14.21)$$

Hence $a(k)$ is analytic in the upper half-plane. By plugging (14.16) into (14.21) one gets

$$\begin{aligned} a \rightarrow \frac{1}{2ik} \left\{ ik \left(1 - \frac{1}{2ik} \int_{-\infty}^x u(y) dy + \dots \right) + \left(1 - \frac{1}{2ik} \int_x^{\infty} u(y) dy \right) \right\} \\ = 1 - \frac{1}{4k} \int_{-\infty}^{\infty} u(y) dy + \dots \end{aligned} \quad (14.22)$$

The scattering amplitude $r(k)$ is defined as follows

$$r(k) = \frac{a(k)}{b(k)}.$$

Also we define $d(k) = \frac{1}{a(k)}$ -amplitude of penetration through the potential barrier. From (14.20) we obtain

$$|r(k)|^2 + |d(k)|^2 = 1 \quad (14.23)$$

This is the “unitary condition”: By definition the potential $u(x)$ is reflectionless if $r(k) \equiv 0$.

In this case $a(k)$ can be found explicitly from the conditions $|a(k)| = 1$ for real k , $a(-k) = \bar{a}(k)$ $a(k) \rightarrow 1$ $k \rightarrow \infty$; $a(k)$ -analytic in the upper half-plane.

If $a(k)$ has no zeros in upper half-plane then $a(k) \equiv 1$. In virtue of condition $a(-k) = \bar{a}(k)$ all zeros are posed on the imaginary axis. Apparently they are exact eigenvalues \aleph_n . $a(k)$ can be presented as the product

$$a(k) = \prod_{m=1}^n \frac{k - i\aleph_m}{k + i\aleph_m} \quad (14.24)$$

For reflectionless potential function

$$Y(k, x) = \frac{B(k, x)}{a(k)} = A(-k, x) \quad (14.25)$$

Lecture 15

Unchecked. Solution of the inverse scattering problem for the Schrodinger equation

The inverse scattering problem is formulated as follows: Suppose that the reflection coefficient $\eta(k)$, set of eigenvalues κ_n ($n = 1, \dots, N$) and coefficients b_n is known. How to find the potential $U(x)$?

It is enough to reconstruct function $\chi(x)$, defined through conditions

$$\chi(k) = \frac{\varphi(k)}{a(k)} e^{ikx} = \frac{B(k)}{a(k)} \quad \text{Im}k > 0 \quad (15.1)$$

$$\chi(k) = \Psi(k) e^{-ikx} = A(-k) \quad \text{Im}k > 0 \quad (15.2)$$

Indeed according to (14.14) in the lower half-plane

$$\chi \rightarrow 1 + \frac{\chi_1}{-ik} + \dots \quad (15.3)$$

$$\chi_1 = -\frac{1}{2} \int_x^\infty U(y) dy \quad U = +2 \frac{\partial}{\partial x} \chi_1 \quad (15.4)$$

In a general case $\chi(k)$ is a function analytic in both upper and lower half plane. It has simple poles on the upper imaginary half axis and a jump on the real axis. Thus it can be presented in a form

$$\chi(k) = 1 + \sum \frac{\chi_n(x)}{k - i\kappa_n} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho(\xi)}{\xi - k} d\xi \quad (15.5)$$

Here $\chi_n(x)$ are residues in poles and

$$\rho(\xi)\chi^+ - \chi^- \quad (15.6)$$

– jump on the real axis

Equation (??) can be rewritten as follows

$$\frac{1}{a(k)}\varphi(k, x) = \Psi(-k, x) + r(k)\Psi(k, x) \quad (15.7)$$

or

$$\frac{1}{a(k)}B(k) = A(-k) + r(k)e^{2ikx}A(-k) \quad (15.8)$$

Now we see that

$$i\chi_n = \frac{B(i\kappa_n)}{a'(\kappa_n)} \quad \chi_n = -i\frac{\varphi(i\kappa_n)e^{-\kappa_n x}}{a'(\kappa_n)} \quad (15.9)$$

and

$$\rho(\xi) = \eta(k)e^{2ikx}A(k) = r(k)e^{2ikx}\chi(-k) \quad (15.10)$$

Then

$$\varphi(i\kappa_n) = b_n\Psi(i\kappa_n) = b_n\chi(-i\kappa_n)e^{-\kappa_n x} \quad (15.11)$$

Finally:

$$\chi_n = -\frac{ib_n}{a'(\kappa_n)}e^{2\kappa_n x}\chi(-i\kappa_n) \quad (15.12)$$

It is remarkable that

$$-\frac{ib_n}{a'(\kappa_n)} = M_n^2 \quad \text{--- realpositivenumber} \quad (15.13)$$

To prove this statement, we differentiate equation (14.1) by k and put $k = i\kappa_n$

$$\frac{d^2\Psi_k}{dx^2} - \kappa_n^2\Psi_k - U(x)\Psi_k = -2i\kappa_n\Psi(i\kappa_n) \quad (15.14)$$

The other hand $\Psi(i\kappa_n)$ satisfies the equation

$$\frac{d^2\Psi(i\kappa_n)}{dx^2} - \kappa_n^2\Psi_k - U\Psi(i\kappa_n) = 0 \quad (15.15)$$

From (??), (??) we obtain equation

$$\frac{d}{dx}\{\Psi_k, \Psi\} = -2i\kappa_n \Psi^2(i\kappa_n) \quad (15.16)$$

As far as $x \rightarrow +\infty$ $\Psi \rightarrow e^{ikx}$

$$\lim_{x \rightarrow +\infty} \{\Psi_n, \Psi\} = 0 \quad (15.17)$$

Then

$$\lim_{x \rightarrow -\infty} \{\Psi_k, \Psi\} = +2i\kappa_n \int \Psi^2(i\kappa_n x) dx \quad (15.18)$$

Now we differentiate equation (15.16) by k and put again $k = i\kappa_n$. We set

$$\varphi_k = a'(k)\Psi(-k) + b'(k)\Psi(k) + b(k)\Psi(k) \quad (15.19)$$

Let us calculate the Wronskian $\{\varphi_k, \Psi\}$ as $x \rightarrow -\infty$ obv

$$\{\varphi_k, \Psi_k\} = b_n \{\varphi_k, \varphi\} \rightarrow 0 \quad \text{at } x \rightarrow -\infty \quad (15.20)$$

We get

$$a'_k \{\Psi(-k), \Psi(k)\}_{-\infty} + b_n \{\Psi_k, \Psi\}_{-\infty} = 0 \quad (15.21)$$

But Wronskian $\{\Psi(-k), \Psi\} = +2\kappa_n$. It does not depend on x and can be calculated at $x \rightarrow +\infty$ where $\Psi(-k) \rightarrow e^{\kappa_n x}$ $\Psi \rightarrow e^{-\kappa_n x}$

Putting together (??) and (??) we obtain

$$-\frac{ib_n}{a'_n} = \frac{1}{\int_{-\infty}^{\infty} \Psi^2 dx} > 0 \quad (15.22)$$

Hence

$$\frac{1}{M_n^2} = \int_{-\infty}^{\infty} \Psi(i\kappa_n, x)^2 dx \quad (15.23)$$

Now we can accomplish derivation of equation solving the ISP. Equation (??) reads

$$\chi(k) = 1 + i \sum \frac{\chi_n(x)}{k - i\kappa_n} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\bar{r}\xi e^{-2i\xi x} \chi(\xi)}{k + \xi} d\xi \quad (15.24)$$

This equation holds in the whole complex plane.

Then

$$\chi_n = M_n^2 e^{-2\kappa_n x} \left(1 - \sum_{m=1}^N \frac{\chi_m}{\kappa_n + \kappa_m} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\bar{r}\xi e^{-2i\xi x} \chi(\xi)}{-i\kappa_n + \xi} d\xi \right) \quad (15.25)$$

As far as $\chi(\xi)$, χ_n are found

$$U = 2 \frac{d}{dx} \left(\sum \chi_n(x) + \frac{1}{2\pi} \int \bar{r}(\xi) e^{-2i\xi x} \chi(\xi) d\xi \right) \quad (15.26)$$

If $r(\xi) \equiv 0$, we obtain already derived finite system of linear algebraic equations

$$\chi_n + M_n^2 e^{-2\kappa_n x} \sum_{m=1}^N \frac{\chi_m}{\kappa_n + \kappa_m} = M_n^2 e^{-2\kappa_n x} \quad (15.27)$$

Note that in the general case $g_n = \chi_n e^{\kappa_n x} = \frac{\varphi(i\kappa_n)}{a'}$ is eigenfunction of equation (14.1)

$$g_n = \chi_n e^{\kappa_n x} = -\frac{i\phi(i\kappa_n)}{a'(i\kappa_n)} = -\frac{ib_n \Psi(i\kappa_n)}{a'(i\kappa_n)} \quad (15.28)$$

is eigenfunction of equation (14.1) with asymptotics

$$g_n \rightarrow -\frac{i}{a'(i\kappa_n)} e^{\kappa_n x} \quad x \rightarrow -\infty \quad (15.29)$$

$$g_n = -\frac{ib_n}{a'(i\kappa_n)} e^{-\kappa_n x} = M_n^2 e^{-2\kappa_n x} \quad x \rightarrow +\infty \quad (15.30)$$

Now

$$\int g_n^2 dx = -\frac{b_n^2}{(a'(i\kappa_n))^2} \int \Psi^2 dx = \frac{1}{\int \Psi^2 dx} = M_n^2 \quad (15.31)$$

So we have now interpretation of coefficients M_n . This is nothing but L_2 norm of eigenfunction $g_n = \chi_n e^{\kappa_n x}$

Let us calculate now $\frac{\partial \chi}{\partial k} \left(\frac{\partial}{\partial \xi} + i \frac{\partial}{\partial \eta} \right) \chi \quad k = \xi + i\eta$

$$\frac{\partial \chi}{\partial k} = \pi \sum \chi_n \delta(k - i\kappa_n) + i(\chi^+ - \chi^-) \delta(\eta) \quad (15.32)$$

$$\frac{\partial \chi}{\partial k} = \pi \sum_{n=1}^N M_n^2 e^{-2\kappa_n x} \chi(-i\kappa_n) \delta(k - i\kappa_n) + ir(\xi) \chi(-k, x) \delta(\eta) \quad (15.33)$$

Completely in accordance with results of lectures 11-12.

Lecture 16

Unchecked. Triangle representation and Marchenko equation

Let us calculate following integral

$$K(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (A(k) - 1)e^{ik(x-y)} dk \quad (16.1)$$

$A(k)$ is analytic in the upper half-plane and $A(k) - 1 \rightarrow 0$ at $|k| \rightarrow \infty$.

Hence $K(x, y) \equiv 0$ if $x > y$. Now we calculate the integral

$$\int_{-\infty}^{\infty} K(x, y)e^{iqy} dy = \int_x^{\infty} K(x, y)e^{iqy} dy = (A(q) - 1)e^{iqx} \quad (16.2)$$

But $A(q)e^{iqx} = \Psi(q, x)$. Hence

$$\Psi(k, x) = e^{ikx} + \int_x^{\infty} K(x, y)e^{iqy} dy \quad (16.3)$$

Equation (16.3) is the triangle representation for the Jost function $\Psi(q, x)$.

Let us introduce function

$$F(s) = \sum_{n=1}^N M_n^2 e^{-\kappa_n s} + \frac{1}{2\pi} \int_{-\infty}^{\infty} r(\xi)e^{i\xi s} d\xi \quad (16.4)$$

and show that $K(x, y)$ satisfies to the following equation

$$K(x, y) = \int_x^\infty K(x, z)F(z + y)dz + F(x + y) = 0 \quad (16.5)$$

This is famous Marchenko equation. To prove this note that $\Psi(-q, x)$ also has triangle representation

$$\Psi(-k, x) = e^{-ikx} + \int_x^\infty K(x, s)e^{iks} ds$$

From (14.14) we get

$$\frac{\Phi}{a} - e^{-ikx} = \int_x^\infty K(x, y)e^{-iks} ds + r(k) \left[e^{ikx} + \int_x^\infty K(x, x)e^{ikz} dz \right] \quad (16.6)$$

Now we multiply equation (16.6) to $\frac{1}{2\pi}e^{iky}$, $y > x$ and integrate over k .

We get immediately

$$\begin{aligned} & K(x, y) + F_0(x + y) + \int_x^\infty K(x, s)F_0(s + y) \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \left(\frac{\Phi}{a} - e^{-ikx} \right) e^{iky} dk = \frac{1}{2\pi} \int_{-\infty}^\infty \left(\frac{\beta}{a} - e^{-ik(y-x)} \right) dk. \end{aligned} \quad (16.7)$$

Here

$$F(s) = \frac{1}{2\pi} \int_{-\infty}^\infty r(\xi)e^{i\xi s} ds$$

Integral in the right hand can be calculated by residues if $y > x$

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^\infty \left(\frac{\beta}{a} - e^{-ik(y-x)} \right) dx = i \sum e^{-\aleph_n y} \frac{\Phi_n(i\aleph_n)}{a'(i\aleph_n)} = \\ & i \sum e^{-\aleph_n y} \frac{b_n}{a'(i\aleph_n)} \Psi_n(i\aleph_n) = - \sum M_n^2 e^{-\aleph_n y} \Psi_n(i\aleph_n) \end{aligned} \quad (16.8)$$

But

$$\Psi(i\aleph_n) = e^{-\aleph_n x} + \int_x^\infty K(x, z)e^{-\aleph_n z} dz \quad (16.9)$$

Plugging (16.9) into (16.8) we obtain equation (16.5) note that this equation is correct only in the half-plane $y > x$. If $y < x$, $K(x, y) = 0$, but

$$F(x + y) + \int_x^\infty K(x, z)F(z + y)dz \neq 0$$

Thus we cannot use Fourier transform (16.2) to find equation for $A(k)$. Note that from (16.3) one gets ($k \rightarrow \infty$ $Imk > 0$)

$$\Psi(k, x) = e^{ikx} \left(1 - \frac{K(x, x)}{ik} + \dots \right)$$

Comparing with expansion (14.14), one gets

$$\frac{1}{2} \int_x^\infty u(y)dy = K(x, x) \quad u = -2 \frac{d}{dx} K(x, x)$$

We shall seek solution of the Marchenko equation in the following form

$$K(x, y) = \sum h_n(x) e^{-\aleph_n y} + \frac{1}{2\pi} \int_{-\infty}^\infty S(k, y) e^{-iky} dk \quad (16.10)$$

Comparing coefficients before different y -exponents one gets system of equations

$$h_n(x) + M_n^2 e^{-\aleph_n x} + M_n^2 \int_x^\infty K(x, z) e^{-\aleph_n z} dz = 0 \quad (16.11)$$

$$s(k, x) + r(k) e^{ikx} + r(k) \int_x^\infty K(s, z) e^{ikz} dz \quad (16.12)$$

Plugging (16.10) into (16.11), (16.12) one obtains

$$h_n(x) + M_n^2 \sum \frac{e^{-(\aleph_n + \aleph_m)x}}{\aleph_n + \aleph_m} + \frac{M_n^2}{2\pi} \int \frac{s(q, x) e^{(iq - \aleph_n)x}}{-\aleph_n + iq} dq = -M_n^2 e^{-\aleph_n x} \quad (16.13)$$

Equation (16.13) goes to (??) if one puts

$$y_n = -h_n e^{-\aleph_n x} \quad h_n = -y_n e^{-\aleph_n x} \quad (16.14)$$

$$s(k, x) = -r(k) y(-k) e^{ikx} \quad (16.15)$$

One can easily check that equations (??) and (16.12) coincide.

Lecture 17

Unchecked. Equation integrated by the local $\bar{\partial}$ - problem

A very important class of nonlinear wave systems, including the Nonlinear Schrodinger and Sine-Gordon equations could be integrated by the use of the local $\bar{\partial}$ -problem.

Suppose that in the nonlocal $\bar{\partial}$ -problem

$$R(\eta, \bar{\eta}, \lambda, \bar{\lambda}) = R(\lambda, \bar{\lambda})\delta(\eta - \lambda)\delta(\bar{\eta} - \bar{\lambda}) \quad (17.1)$$

Now we have the local $\bar{\partial}$ -problem

$$\frac{\partial \chi}{\partial \lambda} = \chi(\lambda)R(\lambda, \bar{\lambda}) \quad (17.2)$$

In the scalar case one has from (??)

$$\frac{\partial}{\partial \lambda} \ln \chi(\lambda) = R(\lambda, \bar{\lambda}) \quad (17.3)$$

and this problem can be solved in the explicit form

$$\ln \chi(\lambda) = \frac{1}{\pi} \int \frac{R(\eta, \bar{\eta})}{\eta - \lambda} d\eta d\bar{\eta} \quad (17.4)$$

Thereafter we will study only the case where χ , R are $n \times n$ matrices.

Suppose that χ is normalized by the condition

$$\chi \rightarrow 1 \quad \lambda \rightarrow \infty \quad (17.5)$$

Here "1" denotes a unit matrix. Then χ satisfies the integral equation

$$\chi = 1 + \frac{1}{\pi} \int \frac{\chi(\eta)R(\eta, \bar{\eta})}{\eta - \lambda} d\eta d\bar{\eta} \quad (17.6)$$

As before we will assume that this equation has unique solution. It means that solution of equation (??) with zero boundary condition

$$\chi \rightarrow 0 \quad \lambda \rightarrow \infty \quad (17.7)$$

is identically zero. We introduce a pair of commuting differential operators

$$D_1\chi = \frac{\partial}{\partial x_1}\chi + \chi A(\lambda) \quad (17.8)$$

$$D_2\chi = \frac{\partial}{\partial x_2}\chi + \chi B(\lambda) \quad (17.9)$$

Here $A(\lambda)$, $B(\lambda)$ - rational functions on λ , depending on x_1, x_2 . The commutativity condition $[D_1, D_2] = 0$ implies that

$$\frac{\partial A}{\partial x_2} - \frac{\partial B}{\partial x_1} + [A, B] = 0 \quad (17.10)$$

In the simplest case when matrix coefficients of A , B do not depend on x_1, x_2 equation (??) is simplified up to

$$[A(\lambda), B(\lambda)] = 0 \quad (17.11)$$

Commutativity of D_1, D_2 means that the system of equations

$$D_1\varphi = \frac{\partial \varphi}{\partial x_1} + \varphi A(\lambda) = 0 \quad (17.12)$$

$$D_2\varphi = \frac{\partial \varphi}{\partial x_2} + \varphi B(\lambda) = 0 \quad (17.13)$$

In the simplest case of constant coefficients φ can be chosen as follow

$$\varphi = e^{-(A(\lambda)x_1 + B(\lambda)x_2)} \quad (17.14)$$

Suppose now that $R(\lambda, \bar{\lambda}, x_1, x_2)$ depends also on x_1, x_2 obeying the equations

$$\frac{\partial R}{\partial x_1} + RA(\lambda) = A(\lambda)R \quad (17.15)$$

$$\frac{\partial R}{\partial x_2} + RB(\lambda) = B(\lambda)R \quad (17.16)$$

The general common solution of equations (??) is

$$R(\lambda, \bar{\lambda}, x_1, x_2) = \varphi^{-1} R_o(\lambda, \bar{\lambda}) \varphi \quad (17.17)$$

Here $R_o(\lambda, \bar{\lambda})$ - an arbitrary matrix function, independent of x_1, x_2 . To prove this statement one just has to remember that φ^{-1} satisfies the equations:

$$\frac{\partial \varphi}{\partial x_1} + A(\lambda)\varphi^{-1} = 0 \quad (17.18)$$

$$\frac{\partial \varphi}{\partial x_2} + B(\lambda)\varphi^{-1} = 0 \quad (17.19)$$

this is because $\partial \varphi^{-1} = -\varphi^{-1} \partial \varphi \varphi^{-1}$

Our further strategy is following. We claim that in virtue of equations (??) a solution χ of the local $\bar{\partial}$ -problems satisfies the system

$$\frac{\partial \chi}{\partial x_1} + \chi A(\lambda) = u(\lambda)\chi \quad (17.20)$$

$$\frac{\partial \chi}{\partial x_2} + \chi B(\lambda) = v(\lambda)\chi \quad (17.21)$$

$$(17.22)$$

Here $u(\lambda), v(\lambda)$ again are rational functions on λ , with the singularities "no worse" than those of $A(\lambda), B(\lambda)$. A more precise statement is following: Suppose $A(\lambda)$ is presented as

$$A(\lambda) = P(\lambda) + \sum_k A_k \quad (17.23)$$

Here $P(\lambda) = P_n \lambda^n + P_{n-1} \lambda^{n-1} + \dots$ is a polynomial of degree n , while A_k is a sum of fractions, associated with the point $\lambda = \lambda_k$

$$A_k = \frac{A_k^1}{\lambda - \lambda_k} + \dots + \frac{A_k^m}{(\lambda - \lambda_k)^{(m)}} \quad (17.24)$$

we claim that $u(\lambda)$ can be decomposed to the similar partial fractions

$$u(\lambda) = \tilde{P}(\lambda) + \sum \tilde{A}_k \quad (17.25)$$

in (??) $\tilde{P}(\lambda)$ is a polynomial of same degree. Moreover $\tilde{P}_n = P_n$. Then

$$\tilde{P}(\lambda) = P_n \lambda^n + \tilde{P}_{n-1} \lambda^{n-1} + \dots \quad (17.26)$$

and again

$$\tilde{A}_k = \frac{\tilde{A}_k^1}{\lambda - \lambda_k} + \dots + \frac{\tilde{A}_k^m}{(\lambda - \lambda_k)^m} \quad (17.27)$$

Similar statement is valid for $B(\lambda), v(\lambda)$. This fundamental statement can be easily proven in the simplest case when $A(\lambda), B(\lambda)$ are polynomials.

To make a proof we apply operators $D_{1,2}$ to equation (??) and mention that $\frac{\partial A}{\partial \lambda} = \frac{\partial A}{\partial \lambda} = 0$, and operation $D_{1,2}$ and $\frac{\partial}{\partial \lambda}$ are commuting. Then we get

$$\frac{\partial}{\partial \lambda} D_1 \chi = \frac{\partial \chi}{\partial x_1} R + \chi \frac{\partial R}{\partial x_1} + \chi R A(\lambda) \quad (17.28)$$

in virtue of (??) one gets

$$\frac{\partial}{\partial \lambda} D_1 \chi = \left(\frac{\partial \chi}{\partial x_1} + \chi A(\lambda) \right) R = D_1 \chi R \quad (17.29)$$

In the same way

$$\frac{\partial}{\partial \lambda} D_2 \chi = D_2 \chi R \quad (17.30)$$

Equations (??) (??) mean that $D_1 \chi, D_2 \chi$ satisfy the equation (??). However, they have a polynomial-type asymptotic when $\lambda \rightarrow \infty$.

One has

$$\chi = 1 + \frac{\chi_1}{\lambda} + \frac{\chi_2}{\lambda^2} + \dots \quad (17.31)$$

$$\chi_1 = -\frac{1}{\pi} \int \chi(\eta) R(\eta, \bar{\eta}) d\eta d\bar{\eta} \quad (17.32)$$

$$\chi_2 = -\frac{1}{\pi} \int \eta \chi(\eta) R(\eta, \bar{\eta}) d\eta d\bar{\eta} \quad (17.33)$$

$$\dots \quad (17.34)$$

To determine $u(\lambda)$, $v(\lambda)$ one just has to introduce new operators

$$L_1\chi = D_1\chi - u(\lambda)\chi \quad (17.35)$$

$$L_2\chi = D_2\chi - v(\lambda)\chi \quad (17.36)$$

where $u(\lambda)$, $v(\lambda)$ - new polynomials and demand

$$L_1\chi \rightarrow 0 \quad \lambda \rightarrow \infty \quad (17.37)$$

$$L_2\chi \rightarrow 0 \quad (17.38)$$

Actually conditions (??) define the coefficients of u, v in unique way. We illustrate this statement in the next lecture. So far we mention only that one can put

$$\chi = \Psi\varphi \quad (17.39)$$

and make sure that λ satisfies to equations

$$\frac{\partial\Psi}{\partial x_1} = u(\lambda)\Psi \quad (17.40)$$

$$\frac{\partial\Psi}{\partial x_2} = v(\lambda)\Psi \quad (17.41)$$

$$(17.42)$$

Compatibility conditions for (??)

$$\frac{\partial u}{\partial x_2} - \frac{\partial v}{\partial x_1} = [u, v] \quad (17.43)$$

Equations (??) is the Lax pair for equation (??). Let $\Psi = \tilde{\varphi}^{-1}$. Then

$$\tilde{\varphi} = \varphi\chi^{-1} \quad (17.44)$$

This is the "dressing formulae"

Lecture 18

Unchecked. 21

In this lecture we will demonstrate how the scheme, elaborated in lecture 20, works in real situation. We will consider one of the simplest examples, leading to integration of the Nonlinear Schrodinger Equation and it's generalization.

Let

$$A(\lambda) = I\lambda, \quad B(\lambda) = I\lambda^2 \quad (18.1)$$

One can choose

$$U(\lambda) = I\lambda + u, \quad V(\lambda) = I\lambda^2 + v\lambda + w \quad (18.2)$$

Function χ satisfies the equation

$$\frac{\partial \chi}{\partial x_1} + \chi I\lambda = (I\lambda + u)\chi \quad (18.3)$$

$$\frac{\partial \chi}{\partial x_2} + \chi I\lambda^2 = (I\lambda^2 + v\lambda + w)\chi \text{ or} \quad (18.4)$$

$$\frac{\partial \chi}{\partial x_1} - u\chi = \lambda[I\chi] \quad (18.5)$$

$$\frac{\partial \chi}{\partial x_2} - (v\lambda + w)\chi = \lambda^2[I\chi] \quad (18.6)$$

Using the expansion of χ

$$\chi = 1 + \frac{\chi_1}{\lambda} + \frac{\chi_2}{\lambda^2} + \frac{\chi_3}{\lambda^3} + \dots \quad (18.7)$$

one immediately finds

$$-u = [I, \chi_1] \quad (18.8)$$

$$-v = [I, \chi_1] \quad v = u \quad (18.9)$$

$$-v\chi_1 - w = [I, \chi_2] = -u\chi_1 - w \quad (18.10)$$

On the other hand

$$[I, \chi_2] = \frac{\partial \chi_1}{\partial x_1} - u\chi_1 = \frac{\partial \chi_1}{\partial x_1} + [I, \chi_1]\chi_1 \quad (18.11)$$

Comparing (??) and (??) we find

$$w = -\frac{\partial \chi}{\partial x_1} \quad (18.12)$$

From (??) (??) we get following equation

$$\frac{\partial \chi}{\partial x_2} - w\chi = \lambda \frac{\partial \chi}{\partial x_1} \quad (18.13)$$

Plugging (??) into (??) gives

$$\frac{\partial \chi_n}{\partial x_2} - w\chi_n = \frac{\partial \chi_{n+1}}{\partial x_1} \quad (18.14)$$

In particular

$$\frac{\partial \chi_1}{\partial x_2} - w\chi_1 = \frac{\partial \chi_2}{\partial x_1} \quad (18.15)$$

Then

$$\frac{\partial}{\partial x_2}[I, \chi_1] - [I, w\chi_1] = \frac{\partial}{\partial x_1}[I, \chi_2] \quad (18.16)$$

Now using equation (??) (??) we get the closed equation, imposed on χ_1

$$\left[I, \frac{\partial \chi_1}{\partial x_2} \right] - \frac{\partial^2 \chi_1}{\partial x_1^2} - \frac{\partial}{\partial x_1}[I\chi_1]\chi_1 + \left[I, \frac{\partial \chi}{\partial x_1} \right] = 0 \quad (18.17)$$

Note that in (??) I is an arbitrary constant matrix. Let us specify.

To go further, one has to specify I . Let it be a block-diagonal matrix

$$I = \begin{bmatrix} +I_1 & 0 \\ 0 & -I_2 \end{bmatrix} \quad (18.18)$$

$$I_1 = \left[\begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{array} \right] \Bigg\} n \quad I_2 = \left[\begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{array} \right] \Bigg\} m \quad (18.19)$$

Let us denote

$$\chi_1 = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \quad \chi_2 = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \quad (18.20)$$

Here q_{11}, p_{11} - squared matrices of dimension n

q_{22}, p_{22} - squared matrices of dimension m

q_{12}, p_{12} - $n \times m$ rectangular matrices

q_{21}, p_{21} - $m \times n$ rectangular matrices

$$U = -[I, \chi] = 2 \begin{bmatrix} 0 & q_{12} \\ -q_{12} & 0 \end{bmatrix} \quad (18.21)$$

From equations (??) one obtains

$$\frac{\partial}{\partial x_1} q_{11} = 2q_{12}q_{21} \quad (18.22)$$

Equations (??) provide that diagonal terms in (??) are satisfied. Equations for off-diagonal terms read

$$2 \frac{\partial}{\partial x_2} q_{12} - \frac{\partial^2 q_{12}}{\partial x_1^2} + 2 \frac{\partial}{\partial x_1} (q_{12}q_{22}) = 2 \frac{\partial q_{12}}{\partial x_1} q_{22} + 2q_{12}q_{21}q_{12} \quad (18.23)$$

$$-2 \frac{\partial}{\partial x_2} q_{21} - \frac{\partial^2 q_{21}}{\partial x_1^2} - 2 \frac{\partial}{\partial x_1} (q_{21}q_{11}) = -2 \frac{\partial q_{21}}{\partial x_1} q_{11} + 2q_{21}q_{12}q_{21} \quad (18.24)$$

$$(18.25)$$

From (??), (??) one gets immediately

$$-2 \frac{\partial q_{12}}{\partial x_2} - \frac{\partial^2 q_{12}}{\partial x_1^2} - 8q_{12}q_{21}q_{12} = 0 - 2 \frac{\partial q_{21}}{\partial x_2} + \frac{\partial^2 q_{21}}{\partial x_1^2} + 8q_{21}q_{12}q_{21} = 0 \quad (18.26)$$

This is the generalized Nonlinear Schrodinger system. From system (??) one can immediately get equations

$$2 \frac{\partial}{\partial x_2} q_{12}q_{21} = \frac{\partial^2 q_{12}}{\partial x_1^2} q_{21} - q_{12} \frac{\partial^2}{\partial x_1} q_{21} = \frac{\partial}{\partial x_1} \left(\frac{\partial q_{12}}{\partial x_1} q_{21} - q_{12} \frac{\partial q_{21}}{\partial x_1} \right) \quad (18.27)$$

$$2 \frac{\partial}{\partial x_2} q_{21}q_{12} = \frac{\partial}{\partial x_1} \left(\frac{\partial q_{21}}{\partial x_1} - q_{21} \frac{\partial q_{12}}{\partial x_1} \right) \quad (18.28)$$

One can easily check that equations (??) are nothing but diagonal terms in equation (??).

To be sure that this is true, one has to calculate matrix function χ_2 . The off-diagonal elements p_{12}, p_{21} could be found from (??)

$$2p_{12} = \frac{\partial q_{12}}{\partial x_1} + 2q_{12}q_{22} \quad (18.29)$$

$$-2p_{21} = \frac{\partial q_{21}}{\partial x_1} - 2q_{21}q_{11} \quad (18.30)$$

$$(18.31)$$

To find the diagonal elements p_{11}, p_{22} , we note that equation (??) generates following set identities

$$[I, \chi_{n+1}] = \frac{\partial \chi_n}{\partial x_1} - u\chi_n = \frac{\partial \chi_n}{\partial x_1} + [I, \chi_1]\chi_n \quad (18.32)$$

In particular

$$[I, \chi_3] = \frac{\partial \chi_2}{\partial x_1} + [I, \chi_1]\chi_2 \quad (18.33)$$

From (??) one gets

$$\frac{\partial}{\partial x_1} p_{11} = -2q_{12}p_{21} = q_{12} \frac{\partial}{\partial x_1} q_{21} + 2q_{12}q_{21}q_{12} = q_{12} \frac{\partial}{\partial x_1} q_{21} + \frac{\partial q_{11}}{\partial x_1} q_{11} \frac{\partial}{\partial x_1} p_{22} = 2q_{21}p_{12} = q_{21} \frac{\partial}{\partial x_1} q_{12} - 2q_{21}q_{11}q_{12} \quad (18.34)$$

Equation (??) can be rewritten as follow

$$\frac{\partial \chi_1}{\partial x_2} + \frac{\partial \chi_1}{\partial x_1} = \frac{\partial \chi_2}{\partial x_1} \quad (18.35)$$

Off-diagonal elements in (??) coincide with equation (??). By the use of (??), one can see that diagonal elements in (??) are identical to equation (??).

To construct higher constants of motion, one can differentiate equation (??) by x_1

$$\frac{\partial}{\partial x_2} \frac{\partial \chi_n}{\partial x_1} = \frac{\partial}{\partial x_1} \left(w\chi_n + \frac{\partial \chi_{n+1}}{\partial x_1} \right) \quad (18.36)$$

Then use the relation

$$\frac{\partial \chi_n}{\partial x_1} = -[I, \chi_1]\chi_n + [I, \chi_{n+1}] \quad (18.37)$$

Plugging (??) into (??) we get

$$-\frac{\partial}{\partial x_2}[I, \chi_1]\chi_n + [I, \chi_{n+1}] = \frac{\partial}{\partial x_1} \left(w\chi_n + \frac{\partial \chi_{n+1}}{\partial x_1} \right) \quad (18.38)$$

Let us denote

$$P_n = ([I, \chi_1]\chi_n)_{diag} \quad (18.39)$$

$$Q_n = \left(w\chi_n + \frac{\partial \chi_{n+1}}{\partial x_1} \right)_{diag} \quad (18.40)$$

P_n and Q_n – diagonal pairs of corresponding matrix functions. They satisfy the equations

$$\frac{\partial P_n}{\partial x_2} + \frac{\partial Q_n}{\partial x_1} = 0 \quad (18.41)$$

Thus $I_n = \int_{-\infty}^{\infty} P_n dx_1$ look like set of matrix motion integrals.

If χ_n presented in a form

$$\chi_n = \begin{bmatrix} q_{11}^{(n)} & q_{12}^{(n)} \\ q_{21}^{(n)} & q_{22}^{(n)} \end{bmatrix} \quad q_{ij}^{(1)} = q_{ij} \quad (18.42)$$

$$P^{(n)} = 2 \begin{bmatrix} q_{12}q_{21}^n & 0 \\ 0 & -q_{21}q_{12}^n \end{bmatrix} \quad (18.43)$$

Thus

$$P_1 = 2 \begin{bmatrix} q_{12}q_{21} & 0 \\ 0 & -q_{21}q_{12} \end{bmatrix} \quad (18.44)$$

To calculate P_2 we remember that $q_{ij}^{(2)} = p_{ij}$. To calculate Q_1 , we mentioned that

$$Q = Q_1 = \left(w\chi_1 + \frac{\partial \chi_2}{\partial x_1} \right)_{diag} = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} \quad (18.45)$$

The upper diagonal term Q_{11} is

$$Q_{11} = -\frac{\partial q_{11}}{\partial x} q_{11} - \frac{\partial q_{12}}{\partial x} q_{21} + \frac{\partial P_{11}}{\partial x} = q_{12} \frac{\partial}{\partial x_1} q_{21} - \frac{\partial q_{12}}{\partial x_1} q_{21} \quad (18.46)$$

$$Q_{22} = -\frac{\partial q_{22}}{\partial x} q_{22} - \frac{\partial q_{21}}{\partial x} q_{12} + \frac{\partial P_{22}}{\partial x} = q_{21} \frac{\partial}{\partial x_1} q_{12} - \frac{\partial q_{21}}{\partial x_1} q_{12} \quad (18.47)$$

One can see that equation

$$\frac{\partial P_1}{\partial x_2} + \frac{\partial Q_1}{\partial x_1} = 0 \quad (18.48)$$

Coincides with equation (??). Note that if $q_{12}, q_{21} \rightarrow 0$ at $|x_1| \rightarrow \infty$, the same is correct for Q_{11}, Q_{22} . Thus quantities $2q_{12}q_{21}, 2q_{21}q_{12}$ are real constant of motion. On the contrary, P_n are just "formal" constants of motion, because in a general case Q_n does not vanish at $|x_1| \rightarrow \infty$. A pure algebraic way for construction of higher "real" constants of motion is not developed yet. This is an interesting unsolved problem.

Lecture 19

Unchecked. 22

In this lecture we study a more general system integrated by the local $\bar{\partial}$ -problem. Let

$$A(\lambda) = I\lambda, B(\lambda = J\lambda), [I, J] = 0 \quad (19.1)$$

I, J - commuting matrices. Now χ satisfies the equations

$$\frac{\partial \chi}{\partial x_1} + \chi I \lambda = (I \lambda + u) \chi \quad (19.2)$$

$$\frac{\partial \chi}{\partial x_2} + \chi J \lambda = (J \lambda + v) \chi \quad (19.3)$$

From (??), (??) we obtain

$$u = -[I, \chi_1] \quad (19.4)$$

$$v = -[J, \chi_1] \quad (19.5)$$

Equations (??), (??) can be presented as follows

$$\frac{\partial \chi_1}{\partial x_1} + [I, \chi_1] \chi = \lambda [I, \chi] \quad (19.6)$$

$$\frac{\partial \chi_1}{\partial x_2} + [J, \chi_1] \chi = \lambda [J, \chi] \quad (19.7)$$

As far as I, J commute one gets

$$\frac{\partial}{\partial x_1} [J, \chi] - \frac{\partial}{\partial x_2} [I, \chi] + [I, [J, \chi_1] \chi] - [J, [I, \chi_1] \chi] = 0 \quad (19.8)$$

Equation (??) is the "master equation", generating the integrable equation together with all its motion constants. To get the basic equation, we just put χ_1 instead of χ . We get the following equation

$$\frac{\partial}{\partial x_1}[J, \chi_1] - \frac{\partial}{\partial x_2}[I, \chi_1] + [I, [J, \chi_1]\chi_1] - [J, [I, \chi_1]\chi_1] = 0 \quad (19.9)$$

One can simplify this equation, remembering that χ can be presented in a form

$$\chi = \Psi\Phi \quad (19.10)$$

Function Ψ satisfied the equations

$$\frac{\partial \Psi}{\partial x_1} = (I\lambda + u)\Psi \quad (19.11)$$

$$\frac{\partial \Psi}{\partial x_2} = (J\lambda + v)\Psi \quad (19.12)$$

Compatibility condition for equations (??),(??) read

$$\frac{\partial u}{\partial x_2} - \frac{\partial v}{\partial x_1} + [I\lambda + u, J\lambda + v] = 0 \quad (19.13)$$

This condition reads

$$[I, v] = [J, u] \quad (19.14)$$

$$\frac{\partial u}{\partial x_2} - \frac{\partial v}{\partial x_1} + [u, v] = 0 \quad (19.15)$$

Resolving equation (??) according to (??),(??), we end up with equation

$$\frac{\partial}{\partial x_2}[I, \chi_1] - \frac{\partial}{\partial x_1}[J, \chi_2] = [[I, \chi_1], [J, \chi_1]] \quad (19.16)$$

One can easily check that equation (??) is exactly equivalent to equation (??).

Let now

$$A(\lambda) = I\lambda \quad (19.17)$$

$$B(\lambda) = J\lambda^2 \quad (19.18)$$

Same as before

$$u(\lambda) = I\lambda + u \quad (19.19)$$

$$v(\lambda) = J\lambda^2 + v\lambda + w \quad (19.20)$$

Function Ψ satisfies the equations

$$\frac{\partial \Psi}{\partial x_1} = (I\lambda + u)\Psi_1 \quad (19.21)$$

$$\frac{\partial \Psi}{\partial x_2} = (J\lambda^2 + v\lambda + w)\Psi_2 \quad (19.22)$$

Compatibility conditions are

$$\frac{\partial u}{\partial x_2} - \frac{\partial}{\partial x_1}(v\lambda + w) + [I\lambda + u, J\lambda^2 + v\lambda + w] = 0 \quad (19.23)$$

as before

$$[I, v] = [J, u] \quad (19.24)$$

$$-\frac{\partial v}{\partial x_1} + [I, w] + [u, v] = 0 \quad (19.25)$$

$$\frac{\partial u}{\partial x_2} - \frac{\partial w}{\partial x_1} + [u, w] = 0 \quad (19.26)$$

If, as in the previous case $J = I$, $u = v$

$$u = 2 \begin{bmatrix} 0 & -q_{12} \\ q_{21} & 0 \end{bmatrix} \quad w = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \quad (19.27)$$

Equation (??) reduces to the form

$$[i, w] = \frac{\partial u}{\partial x_1} \quad (19.28)$$

it gives

$$w_{12} = -\frac{\partial q_{12}}{\partial x_1} \quad w_{21} = -\frac{\partial q_{21}}{\partial x_1} \quad (19.29)$$

Now

$$[u, w] = 2 \begin{bmatrix} -q_{12}w_{21} - w_{12}q_{21} & -q_{12}w_{22} + w_{11}q_{12} \\ q_{21}w_{11} - w_{22}q_{21} & q_{21}w_{12} + w_{21}q_{12} \end{bmatrix} = \quad (19.30)$$

$$= 2 \begin{bmatrix} \frac{\partial}{\partial x} q_{12}q_{21} & -q_{12}w_{22} + w_{11}q_{12} \\ q_{21}w_{11} - w_{22}q_{21} & q_{21}w_{12} + w_{21}q_{12} \end{bmatrix} \quad (19.31)$$

Diagonal parts of equation (??) gives

$$-\frac{\partial w_{11}}{\partial x_1} - 2\frac{\partial}{\partial x}q_{12}q_{21} = 0 \quad w_{11} = 2q_{12}q_{21} \quad (19.32)$$

similarly

$$w_2 = -2q_{21}q_{12} \quad (19.33)$$

Finally

$$w = \begin{bmatrix} 2q_{12}q_{21} & -\frac{\partial q_{12}}{\partial x_1} \\ -\frac{\partial q_{21}}{\partial x_1} & -2q_{21}q_{12} \end{bmatrix} \quad (19.34)$$

off-diagonal parts in (??) give equations

$$-2\frac{\partial q_{12}\partial x_2}{+} \frac{\partial^2 q_{12}}{\partial x_1^2} + 8q_{12}q_{21}q_{12} = 0 \quad (19.35)$$

$$+2\frac{\partial q_{21}\partial x_2}{+} \frac{\partial^2 q_{12}}{\partial x_1^2} + 8q_{21}q_{12}q_{21} = 0 \quad (19.36)$$

which are identical to equations (??). So far x_1, x_2 were arbitrarily complex variables. Let us assume

$$\frac{\partial}{\partial x_2} = i\frac{\partial}{\partial t} \quad \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x} \quad (19.37)$$

Equations (??) read

$$-2i\frac{\partial q_{12}}{\partial t} + \frac{\partial^2 q_{12}}{\partial x^2} + 8q_{12}q_{21}q_{12} = 0 \quad (19.38)$$

$$-2i\frac{\partial q_{21}^\dagger}{\partial t} + \frac{\partial^2 q_{21}^\dagger}{\partial x^2} + 8q_{21}^\dagger q_{12}^\dagger q_{21}^\dagger = 0 \quad (19.39)$$

The most important special case of system (??) is following. Suppose $n = 1$, $m = N$ then

$$q_{12} = (q_1 \dots q_n) \quad q_{21} = \begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix} \quad (19.40)$$

$$q_{12}q_{21} = u = \sum_{k=1}^N p_k q_k \quad (19.41)$$

Equations (??) read

$$-2i\frac{\partial q_k}{\partial t} + \frac{\partial^2 q_k}{\partial x_2} + 8uq_k = 0 \quad (19.42)$$

$$-2i\frac{\partial \bar{p}_k}{\partial t} + \frac{\partial^2 \bar{p}_k}{\partial x_2} + 8\bar{u}\bar{p}_k = 0 \quad (19.43)$$

Now assume that $\bar{p}_k = \alpha q_k$ $\alpha = \pm 1$. Then u is real

$$u = \sum_{k=1}^N \alpha_k |q_k|^2 \quad (19.44)$$

We call system (??) generalized Manakov system.

Lecture 20

Unchecked. 23

Let us restate where we are:

Function χ satisfies the equations

$$i \frac{\partial \chi}{\partial t} - (u\lambda + w)\chi = \lambda^2 [I, \chi] \quad I = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{block-diagonal matrix} \quad (20.1)$$

at $\lambda \rightarrow \infty$

$$\chi = 1 + \frac{\chi_1}{\lambda} + \frac{\chi_2}{\lambda^2} + \dots \quad (20.3)$$

$$\chi = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \quad u = 2 \begin{bmatrix} 0 & q_{12} \\ -q_{21} & 0 \end{bmatrix} \quad (20.4)$$

$$q_{11} = 2\partial^{-1} q_{12} q_{21} \quad q_{22} = -2\partial^{-1} q_{21} q_{12} \quad (20.5)$$

$$w = \frac{\partial \chi}{\partial x} = \begin{bmatrix} 2q_{12} q_{21} & q_{12x} \\ q_{21x} & -2q_{21} q_{12} \end{bmatrix} \quad (20.6)$$

off-diagonal matrices q_{12}, q_{21} satisfy the equations

$$2i \frac{\partial q_{12}}{\partial t} - \frac{\partial^2 q_{12}}{\partial x^2} - 8q_{12} q_{21} q_{12} = 0 \quad (20.7)$$

$$2i \frac{\partial q_{21}}{\partial t} + \frac{\partial^2 q_{21}}{\partial x^2} + 8q_{21} q_{12} q_{21} = 0 \quad (20.8)$$

assuming that $n = 1$ $m = N$, we choose I of the form

$$I = \left[\begin{array}{ccc} 1 & & 0 \\ & -1 & \\ & & \ddots \\ 0 & & & -1 \end{array} \right] \Bigg\} N+1 \quad q_{12} = (q_1 \dots q_n) \quad p_{21} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \quad (20.9)$$

$$u = \sum_{k=1}^N q_k \varphi_k \quad (20.10)$$

Equations (??) read

$$2iq_{kt} - q_{kxx} - 8uq_k = 0 \quad (20.11)$$

$$2ip_{kt} + p_{kxx} + 8uq_k = 0 \quad (20.12)$$

$$(20.13)$$

We assume that

$$p_k = \alpha_k \bar{q}_k \quad \alpha_k = \pm 1 \quad (20.14)$$

Then u is real

$$u = \sum_{k=1}^N \alpha_k |q_k|^2 \quad (20.15)$$

Now system (??) reduces to smaller systems

$$2iq_{kt} - q_{kxx} - 8uq_k = 0 \quad (20.16)$$

This system has a trivial solution

$$q_k = a_k e^{-4iu_0 t} \quad (20.17)$$

$$u_0 = \sum_{k=1}^N \alpha_k |q_k|^2 \quad (20.18)$$

Solution (??)-(??) is the "condensate". Let us assume

$$N = N_1 + N_2 \quad \begin{array}{l} \alpha_k = 1 \quad 1 \leq k \leq N_1 \\ \alpha_k = -1 \quad N_1 + 1 < k \leq N_2 \end{array} \quad (20.19)$$

Now equation (??) is a generalized Manakov system, partly focusing ($1 \leq k \leq N_1$), partly defocusing. One can study stability of the condensate. By performing the transform

$$q_k \rightarrow q_k e^{-4iu_0 t} \quad (20.20)$$

one transforms (??) to the form

$$2iq_{kt} - q_{kxx} - 8(u - u_0)q_k = 0 \quad (20.21)$$

Assuming that $q_k = A_k e^{-i\phi_k}$ we derive system (??) to the form

$$\begin{cases} \frac{\partial}{\partial t} A_k^2 + \frac{\partial}{\partial x} (A_k^2 \phi_k') = 0 \\ A_k = \left(\frac{\partial}{\partial t} \phi_k + \frac{1}{2} \phi_k'^2 \right) - 4(u - u_0)A_k + \frac{1}{2} A_{kxx} = 0 \end{cases} \quad (20.22)$$

Assuming that $A_k = a_k + \delta A_k$ one can linearize the system (??),(??). The result of linearization is following system

$$2 \frac{\partial}{\partial t} \delta A_k + q_k \frac{\partial^2 \phi_k}{\partial x^2} = 0 \quad (20.23)$$

$$q_k \frac{\partial \phi_k}{\partial t} - 4q_k \delta u + \frac{1}{2} \frac{\partial^2}{\partial x^2} \delta A_k = 0 \quad (20.24)$$

This system can be rewritten as follows

$$2 \frac{\partial^2}{\partial t^2} \delta A_k + \frac{\partial^2}{\partial x^2} \left(4q_k \delta u - \frac{1}{2} \frac{\partial^2}{\partial x^2} \delta A_k \right) = 0 \quad (20.25)$$

Here $\delta u = 2 \sum \alpha q_k \delta A_k$. By multiplication by $\alpha_k q_k$ and summation we obtain closed equation for δu

$$\frac{\partial^2}{\partial t^2} \delta A_k + \frac{\partial^2}{\partial x^2} \left(4u_0 \delta u - \frac{1}{4} \frac{\partial^2}{\partial x^2} \delta u \right) = 0 \quad (20.26)$$

Assuming that $\delta u \approx e^{i\Omega t + ipx}$ we get

$$\Omega^2 = -4u_0 p^2 + \frac{1}{4} p^4 \quad (20.27)$$

Condensate is stable if $u_0 < 0$ – (defocusing part is prevailing) and unstable if $u_0 > 0$.

In the Manakov case

$$I = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{bmatrix} \quad u = 2 \begin{bmatrix} 0 & q_1 & \dots & q_n \\ -p_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \\ -p_n & 0 & & 0 \end{bmatrix} \quad w = \begin{bmatrix} 2u & q_{1x} & \dots & q_{nx} \\ p_{1x} & -2p_1 q_1 & \dots & -2p_1 q_n \\ p_{2x} & -2p_2 q_1 & \dots & -2p_2 q_n \\ \vdots & \vdots & & \vdots \\ p_{nx} & -2p_n q_1 & \dots & -2p_n q_n \end{bmatrix} \quad (20.28)$$

Let us present the solution of equation (??) in a form

$$\chi = \Psi\Phi \quad (20.29)$$

Here Φ is a common solution of the system

$$\frac{\partial\Phi}{\partial x} + \Phi I\lambda = 0 \quad (20.30)$$

$$i\frac{\partial\Phi}{\partial t} + \Phi I\lambda^2 = 0 \quad (20.31)$$

Φ is the fundamental solution of the following system

$$\begin{cases} \frac{\partial\Psi_1}{\partial x} = \lambda\Psi_1 + 2\sum_{k=1}^n q_k\Psi_{k+1} \\ \frac{\partial\Psi_{k+1}}{\partial x} = -\lambda\Psi_{k+1} - 2p_k\Psi_1 \end{cases} \quad (20.32)$$

$$i\frac{\partial\Psi_1}{\partial t} = (\lambda^2 + 2u)\Psi_1 + \sum(2\lambda q_k + q'_k)\Psi_{k+1} \quad (20.33)$$

$$i\frac{\partial\Psi_{k+1}}{\partial t} = -\lambda^2\Psi_{k+1} + (-2\lambda p_k + p'_k)\Psi_1 - 2p_k\sum_{l=1}^n q_l\Psi_{l+1} \quad (20.34)$$

In the condensate case equations (??) and (??) can be easily solved. Now p_k, q_k do not depend on x . Let us denote

$$v = \sum_{k=1}^n q_k\Psi_{k+1} \quad (20.35)$$

From (??) we get

$$\frac{\partial\Psi_1}{\partial x} = \lambda\Psi_1 + 2v \quad (20.36)$$

$$\frac{\partial v}{\partial x} = -\lambda v - 2u\Psi_1 \quad (20.37)$$

now we study time-dependent solution. Equations (??) transform now to the form

$$i\frac{\partial\Psi_1}{\partial t} = (\lambda^2 + 2u)\Psi_1 + 2\lambda v \quad (20.38)$$

$$i\frac{\partial\Psi_{k+1}}{\partial t} = -\lambda^2\Psi_{k+1} - 2\lambda p_k\Psi_1 - 2p_k v \quad (20.39)$$

From (??) one get

$$i\frac{\partial V}{\partial t} - (\lambda^2 + 2u)V - 2\lambda u\Psi_1 = 0 \quad (20.40)$$

Now remember that

$$q_k = a_k e^{-4iut} \quad p_k = b_k e^{4iut} \quad (20.41)$$

Assuming that

$$\Psi_{k+1} = \varphi_{k+1} e^{2iut} \Psi_1 = \varphi_1 e^{-2iut} \quad (20.42)$$

We can exclude time-dependance from. Indeed:

$$V = V_0 e^{-2iut} \quad (20.43)$$

and equations (??) take form:

$$\begin{cases} i\frac{\partial a_1}{\partial t} = \lambda^2 a_1 + 2\lambda V_0 \\ i\frac{\partial V_0}{\partial x} = -\lambda V_0 - 2ua_1 \end{cases} \quad (20.44)$$

systems (??) (??) are compatible. One can put

$$a_k = e^{Qx - \lambda k t} \xi_k \quad V_0 = \sum_{k=1}^n a_k \xi_{k+1} \quad (20.45)$$

From (??) we get the same $k = 1$ algebraic equations for ξ

$$(q - \lambda)\xi_1 + 2V_0 = 0 \quad (20.46)$$

$$(q + \lambda)V_0 = -2u\xi_1 \quad (20.47)$$

From (??) we get

$$q^2 = \lambda^2 - 2u \quad q_{12} = \pm\sqrt{\lambda^2 - 2u} \quad (20.48)$$

$q_{12} = \pm\sqrt{\lambda^2 - 2u}$ are not only eigenvalues of this system. There is also degenerate solution $q = -\lambda$. Now $\xi_1 = 0$ $V_0 = 0$, ξ_k are arbitrary constants, satisfying the condition

$$\sum_{k=1}^n \xi_k a_k = 0 \quad (20.49)$$

Equation (??) has $n - 1$ parametrized family of solutions, so eigenvalue $q = -1$ is $n - 1$ time degenerated. Totally we have $n + 1$ linearly independent solutions. One eigenvalue behaves at $\lambda \rightarrow \infty$ as λ , all other behave like $-\lambda$. Hence, we can construct a fundamental solution

$$\Psi \rightarrow e^{+I\lambda + iI\lambda^2} \quad \lambda \rightarrow \infty \quad (20.50)$$

as far as $\Phi = e^{-I\lambda - iI\lambda^2} \chi \rightarrow 1$ as $\lambda \rightarrow \infty$. This is what we need. Note that the inverse matrix $\tilde{\Phi} = \Psi^{-1}$ satisfies the equations

$$\frac{\partial \tilde{\Phi}}{\partial x} + \tilde{\Phi}(I\lambda + u) = 0 \quad (20.51)$$

$$i \frac{\partial \tilde{\Phi}}{\partial x} + \tilde{\Phi}(I\lambda^2 + u\lambda w) = 0 \quad (20.52)$$

and can be used for dressing. It is especially interesting to study dressing around the condensate background.

Lecture 21

Unchecked. Solitonic solution of equation integrable by local \bar{D} -problems

Let us formulate the following problem from linear algebra. Let χ be a $M \times M$ complex-valued matrix. $\chi(\lambda)$ – a rational matrix-valued function on complex plane λ . Suppose that all poles of $\chi(\lambda)$ are simple and it can be decomposed into partial fractions

$$\chi = 1 + \sum_{n=1}^N \frac{A_n}{\lambda - a_n} \quad (21.1)$$

Here a_n – some complex numbers.

Let χ^{-1} – inverse matrix. It is a rational function again. Its poles are zeros of the rational function $\det \chi$. In general case numerator of this function is a polynomial of degree $N \times M$. This in general position inverse function χ^{-1} has $N \times M$ poles.

In this chapter we will study following class of rational functions

$$\chi^{-1} = 1 + \sum_{n=1}^N \frac{B_n}{\lambda - b_m} \quad a_n \neq b_m \quad (21.2)$$

Certainly it is a very special class of rational functions. First let $N = 1$. The function is

$$\chi = 1 + \frac{A}{\lambda - a} \quad (21.3)$$

$$\chi^{-1} = 1 + \frac{B}{\lambda - b} \quad (21.4)$$

the condition $\chi\chi^{-1} = 1$ is resolved if

$$A \left(1 + \frac{B}{a - b} \right) = 0; \quad \left(1 + \frac{1}{b - a} A \right) B = 0 \quad (21.5)$$

Let us introduce $A = (a - b)P$, $B = -(a - b)\tilde{P}$. Equation () now reads

$$P(1 - \tilde{P}) = 0; \quad (1 - P)\tilde{P} = 0 \quad (21.6)$$

From equation () we get $\tilde{P} = P$ and $P = P^2$, therefore P is the projective operator and

$$\chi = 1 + \frac{a - b}{\lambda - a} P \quad \chi^{-1} = \left(1 - \frac{a - b}{\lambda - b} P \right) \quad (21.7)$$

Function χ , χ^{-1} could be found in form of products of the ‘‘Blaschke factors’’

$$\chi = \left(1 + \frac{P_1}{\lambda - a_1} \right) \dots \left(1 + \frac{P_n}{\lambda - a_n} \right) \quad (21.8)$$

$$\chi^{-1} = \left(1 - \frac{P_n}{\lambda - b_n} \right) \left(1 - \frac{P_{n-1}}{\lambda - b_{n-1}} \right) \dots \left(1 - \frac{P_1}{\lambda - b_1} \right) \quad (21.9)$$

This representation is of course not unique. One can put factors including fraction $\frac{1}{\lambda - a_k}$ in an arbitrary order.

Another solution of the same problem can be constructed as follows: as far as a_n , b_n all are different, function

$$P_n = \chi|_{\lambda=b_n} \quad Q_n = \chi^{-1}|_{\lambda=a_n} \quad (21.10)$$

From the condition $\chi\chi^{-1} = 1$ one get

$$A_n Q_n = 0 \quad P_n B_n = 0 \quad (21.11)$$

From the condition $\chi^{-1}\chi = 1$ one obtains

$$B_n P_n = 0 \quad Q_n A_n = 0 \quad (21.12)$$

of course conditions () follow from (). From (),() we conclude that all residues A_n , B_n are degenerate matrices.

Equations () () read

$$A_n \left(1 + \sum_{m=1}^N \frac{B_m}{a_n - b_m} \right) = 0 \quad \left(1 + \sum_{n=1}^N \frac{A_m}{b_n - a_m} \right) B_n = 0 \quad (21.13)$$

These equations could be turned into linear systems. We will do this in the simplest possible case when all matrices are rank one and can be presented as tensor products of two vectors

$$A_n = \lambda_n \oplus P_n \quad B_n = q_n \oplus \mu_n \quad (21.14)$$

In other words

$$A_{n\alpha\beta} = \lambda_{n\alpha} P_{n\beta} \quad B_{n\alpha\beta} = q_{n\alpha} \mu_{n\beta} \quad (21.15)$$

We will assume that $P_{n\alpha}$, $q_{n\alpha}$ are known while λ_n , μ_n are unknown. Let us denote

$$V_{nm} = \sum_{\gamma} P_{n\gamma} q_{\gamma m} \quad (21.16)$$

(One can equation () are equivalent)

$$P_n + \sum_{n=1}^N \frac{V_{nm} \mu_m}{a_n - b_n} = 0 \quad (21.17)$$

$$q_n + \sum_{n=1}^N \frac{\lambda_m V_{nm}}{a_n - b_n} = 0 \quad (21.18)$$

$$(21.19)$$

When A_n , B_n are found

$$\chi_1 = \sum_{n=1}^N A_n \quad (21.20)$$

If $N + 1$ equations () () are simple. Now $V = P_{\gamma} q_{\gamma}$ - a number

$$\mu = -\frac{a-b}{V} P \quad \lambda = \frac{a-b}{V} q$$

so

$$A = (a-b) \frac{q \oplus p}{V} \quad (21.21)$$

$$B = -(a-b) \frac{q \oplus p}{V} \quad (21.22)$$

Apparently $\frac{q \oplus p}{V} = \mathbf{P}$ – is the projective operator. We obtained previous result. To find the equation imposed on P, Q we consider equations

$$\frac{\partial \chi}{\partial x_1} + \chi A(\lambda) = U \chi \quad (21.23)$$

$$\frac{\partial \chi}{\partial x_2} + \chi B(\lambda) = V \chi \quad (21.24)$$

Inverse matrix satisfy the equation

$$-\frac{\partial \chi^{-1}}{\partial x_1} + A(\lambda) \chi^{-1} = \chi^{-1} U(\lambda) \quad (21.25)$$

$$-\frac{\partial \chi^{-1}}{\partial x_2} + B(\lambda) \chi^{-1} = \chi^{-1} V(\lambda) \quad (21.26)$$

Then

$$\left(\frac{\partial \chi}{\partial x_1} + \chi A \right) \chi^{-1} = U = \chi \left(-\frac{\partial \chi^{-1}}{\partial x_1} + A(\lambda) \chi^{-1} \right) \quad (21.27)$$

$$\left(\frac{\partial \chi}{\partial x_2} + \chi B \right) \chi^{-1} = V = \chi \left(-\frac{\partial \chi^{-1}}{\partial x_2} + B(\lambda) \chi^{-1} \right) \quad (21.28)$$

Neither U , nor V have singularities at $\lambda = a_n, b_n$. This imposes the following equation on vectors P and q

$$\begin{cases} \frac{\partial P_n}{\partial x_1} + P_n A = 0 \\ \frac{\partial P_n}{\partial x_2} + P_n B = 0 \end{cases} \quad \begin{cases} \frac{\partial q}{\partial x_1} - A q = 0 \\ \frac{\partial q}{\partial x_2} - B q = 0 \end{cases} \quad (21.29)$$

Equations are satisfied if

$$P_n = P_n^{(0)} \Phi_n \quad q_n = \Phi_n^{-1} q_n^{(0)} = S_n q_n^{(0)} \quad (21.30)$$

Here $\Phi_n = \Phi|_{\lambda=a_n}$, $S_n = \Phi^{-1}|_{\lambda=b_n}$. The next question we would like to discuss here is a problem of reduction. Suppose that J is a matrix with a constant elements, satisfying the condition $J^2 = 1$ and commuting with A and B

$$[J, A(\lambda)] = 0 \quad [J, B(\lambda)] = 0 \quad (21.31)$$

Let kernel of local \bar{D} - problem satisfy the condition

$$R^\dagger(\bar{\lambda}, \lambda) = -JR(\lambda, \bar{\lambda})J \quad R(\lambda, \bar{\lambda}) = -JR^\dagger(\bar{\lambda}, \lambda)J \quad (21.32)$$

Now compare equation for $\chi^\dagger(\bar{\lambda}, \lambda)$ and $\chi^{-1}(\lambda, \bar{\lambda})$

$$\frac{\partial}{\partial \bar{\lambda}} \chi^\dagger(\bar{\lambda}, \lambda) = R^\dagger(\bar{\lambda}, \lambda) \chi^\dagger(\bar{\lambda}, \lambda) \frac{\partial}{\partial \bar{\lambda}} \chi^{-1}(\lambda, \bar{\lambda}) = -R(\bar{\lambda}, \lambda) \chi^{-1}(\lambda, \bar{\lambda}) \quad (21.33)$$

From () one gets

$$\frac{\partial}{\partial \bar{\lambda}} J \chi^\dagger(\bar{\lambda}, \lambda) J = J R^\dagger(\bar{\lambda}, \lambda) J \cdot J \chi^\dagger(\bar{\lambda}, \lambda) J = -R(\lambda, \bar{\lambda}) J \chi^\dagger(\bar{\lambda}, \lambda) J \quad (21.34)$$

Comparing () and () one concludes

$$\chi^{-1} J \chi^\dagger(\bar{\lambda}, \lambda) J \quad (21.35)$$

If this condition is satisfied

$$b_n = \bar{a}_n \quad B_n = J A_n^\dagger J \quad (21.36)$$

if $A_n = \lambda_n \oplus P_n$ and $B_n = q_n \oplus \mu_n$ the equation () means

$$q_n = J \bar{P}_n \quad \mu_n = \bar{\lambda}_n J \quad (21.37)$$

Therefore we replace $\lambda \rightarrow i\lambda$. Hence

$$\chi \rightarrow 1 + \frac{\chi_1}{i\lambda} \quad (21.38)$$

$$\chi^\dagger(\bar{\lambda}) \approx 1 - \frac{\chi^\dagger}{i\lambda} \quad (21.39)$$

$$\chi^{-1} \approx 1 - \frac{\chi_1}{i\lambda} \quad (21.40)$$

and reduction means

$$x i_1^\dagger = J \chi_1 J \quad J^2 = 1 \quad (21.41)$$

Remembering that

$$\chi_1 = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \quad (21.42)$$

we choose

$$J = \begin{bmatrix} 1 & 0 \\ 0 & \Lambda \end{bmatrix} \quad \Lambda^2 = 1 - \text{diagonal matrix } n \times n \quad (21.43)$$

now

$$J \chi_1 J = \begin{bmatrix} q_{11} & q_{12} \Lambda \\ \Lambda q_{21} & \Lambda q_{22} \Lambda \end{bmatrix} = \begin{bmatrix} q_{11}^\dagger & q_{21}^\dagger \\ q_{12}^\dagger & q_{22}^\dagger \end{bmatrix} \quad (21.44)$$

In other words

$$q_1 1 = q_1 1^\dagger, \quad q_{12} \Lambda = q_{21}^\dagger, \quad \Lambda q_{21} = q_{12}^\dagger, \quad q_{22}^\dagger = \Lambda q_{22} \Lambda, \quad q_{21} = \Lambda q^\dagger q_{12}^\dagger \quad (21.45)$$

and equations for q_{12} , q_{21} reduce to the single equation

$$2i \frac{\partial q_{12}}{\partial x} + Q_{12\lambda x} + 8q_{12} \Lambda q_{12}^\dagger q_{12} = 0 \quad (21.46)$$

If $\Lambda = 1$ this is “pure focusing” equation

If $\Lambda = -1$ this is “pure defocusing” equation

For general diagonal matrix Λ equations are “mixed”.

Let $q_{12} = 1/2q$ and q is square matrix, satisfying the equation

$$i \frac{\partial q}{\partial t} + \frac{1}{2} \frac{\partial^2 q}{\partial x^2} + q \Lambda q^\dagger q = 0 \quad (21.47)$$

this equation allows further reductions. for instance one can assume that q is either symmetric or antisymmetric matrix. More generally

$$q^{tr} = \pm |q| \quad [L, \Lambda] = 0 \quad (21.48)$$

Different reductions in () make possible to construct (...) of systems of the NLSE. For instance if

$$q = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad q^\dagger = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{b} & \bar{c} \end{bmatrix} \quad \Lambda = 1 \quad (21.49)$$

$$qq^\dagger = \begin{bmatrix} |a|^2 + |b|^2 & a\bar{b} + b\bar{c} \\ b\bar{a} + c\bar{b} & |b|^2 + |c|^2 \end{bmatrix} \quad (21.50)$$

We have the following system:

$$ia_t + \frac{1}{2} a_{xx} + (|a|^2 + 2|b|^2)a + b^2 \bar{c} = 0 \quad (21.51)$$

$$ib_t + \frac{1}{2} b_{xx} + (|a|^2 + |b|^2 + |c|^2)b + a\bar{b}c = 0 \quad (21.52)$$

$$ic_t + \frac{1}{2} c_{xx} + (2|b|^2 + |c|^2)c + b^2 \bar{a} = 0 \quad (21.53)$$

$$(21.54)$$

Further reduction $a = c$ reduces system () to two equations

$$ia_t + \frac{1}{2}a_{xx} + (|a|^2 + 2|b|^2)a + b^2\bar{a} = 0 \quad (21.55)$$

$$ib_t + \frac{1}{2}b_{xx} + (2|a|^2 + |b|^2)b + a^2\bar{b} = 0 \quad (21.56)$$

$$(21.57)$$

In fact this system is equivalent to the system of two independent NLSE for

$$\Psi^\pm = \frac{1}{2}(a \pm b) \quad (21.58)$$

Other simple system that deserves to be studied

$$q_1 = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \quad q_2 = \begin{bmatrix} a & b & c \\ b & d & b \\ c & b & a \end{bmatrix} \quad q_3 = \begin{bmatrix} a & b & 0 \\ b & 0 & -b \\ 0 & -b & -a \end{bmatrix} \quad (21.59)$$

Lecture 22

Unchecked. Simple solitonic solution for the generalized Manakov System

Let

$$A(\lambda) = iI\lambda \quad B(\lambda) = -I\lambda^2 \quad (22.1)$$

Function Φ satisfies the equations

$$\frac{\partial \Phi}{\partial x} + i\Phi I\lambda = 0 \quad (22.2)$$

$$i\frac{\partial \Phi}{\partial t} - \Phi I\lambda^2 = 0 \quad (22.3)$$

$$\Phi = e^{-iI(\lambda x + \lambda^2 t)} \quad (22.4)$$

$$S = e^{iI(\lambda x + \lambda^2 t)} \quad (22.5)$$

In future we assume that some reduction is performed so $b_n = \bar{a}_n$. Suppose we have only one pole $a_1 = a = \xi + i\eta$, $b_1 = \bar{a}$. Now assume that dimension of space is $N + 1$, $N = n + m$. Let

$$\begin{aligned} \vec{q}^{(0)} &= (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1}, \dots, \xi_{n+m}) \\ p_0 &= (\bar{\xi}_0, \bar{\xi}_1, \dots, \bar{\xi}_n, -\bar{\xi}_{n+1}, \dots, -\bar{\xi}_{n+m}) \\ q &= (q_0, q_1, \dots, q_n, q_{n+1}, \dots, q_{n+m}) \\ p &= (\bar{q}_0, \bar{q}_1, \dots, \bar{q}_n, -\bar{q}_{n+1}, \dots, -\bar{q}_{n+m}) \end{aligned}$$

$$\begin{aligned}
q_0 &= \xi_0 e^{i(ax+a^2t)} \\
q_k &= \xi_k e^{-i(ax+a^2t)} \\
p_0 &= \bar{\xi}_0 e^{-i(\bar{a}x+\bar{a}^2t)} \\
p_k &= \bar{\xi}_k e^{+i(\bar{a}x+\bar{a}^2t)} \quad 1 \leq k \leq n \\
p_0 &= -\bar{\xi}_k e^{+i(\bar{a}x+\bar{a}^2t)} \quad n+1 \leq k \leq n+m
\end{aligned}$$

$$V = (p, q) = |\xi_0|^2 e^{i(a-\bar{a})x+i(a^2-\bar{a}^2)t} + W e^{-i(a-\bar{a})x+i(a^2-\bar{a}^2)t} \quad (22.6)$$

$$W = \sum_{k=1}^n |\xi_k|^2 - \sum_{k=n+1}^{n+m} |\xi_k|^2 \quad (22.7)$$

V is real function

$$V = |\xi_0|^2 e^{-2\eta(x+2\xi t)} + W e^{2\eta(x+2\xi t)} \quad (22.8)$$

Solution is regular, if $W > 0$! According (25.14)

$$\lambda = \frac{a - \bar{a}}{v} q = \frac{2i\eta}{v} q$$

remeber that $q_{12} = q_1 \dots q_n$. For q_k one sets

$$q_k = \frac{2i\eta}{V} \xi_0 \xi_k e^{i(a+\bar{a})x+i(a^2+\bar{a}^2)t} \quad (22.9)$$

$$q_k = \frac{2i\eta \bar{\xi}_0 \bar{\xi}_k e^{2i(\xi x + (\xi^2 - \eta^2)t)}}{|\xi_0|^2 e^{-2\eta(x+2\xi t)} + W e^{2\eta(x+2\xi t)}} \quad (22.10)$$

One can put $\xi_k = \xi_0 \Psi_k$. Then ξ_0 drops out of the equation. Solution is defined by a position of eigenvalue $a = \xi + i\eta$ and by the set of complex parameters Ψ_k . We showed that soliton does exist only if

$$W > 0 \quad \sum_{k=1}^n |\Psi_k|^2 > \sum_{k=n+1}^{n+m} |\Psi_k|^2$$

The ‘‘focusing’’ part showed prevails over the ‘‘defocusing’’. Of course this solution could be obtained by pure elementary methods.