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Kolmogorov spectra of weak turbulence in media with two types of interacting waves

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Abstract

A system of one-dimensional equations describing media with two types of interacting waves is considered. This system can be viewed as an alternative to the model introduced by Majda, McLaughlin and Tabak in 1997 for assessing the validity of weak turbulence theory. The predicted Kolmogorov solutions are the same in both models. The main difference between both models is that coherent structures such as wave collapses and quasisolitons cannot develop in the present model. As shown recently these coherents structures can influence the weakly turbulent regime. It is shown here that in the absence of coherent structures weak turbulence spectra can be clearly observed numerically. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Weak turbulence theory is an efficient tool for describing turbulence in systems dominated by resonant interactions between small-amplitude waves. One of the key ingredients to the theory of weak wave turbulence is the so-called Kolmogorov spectrum [1]. Kolmogorov weak-turbulence spectra have been observed in several physical systems (for example, in a sea of wind-driven, weakly coupled, dispersive water waves). We believe that Kolmogorov weak-turbulence spectra are a useful theoretical tool for explaining various complex wave phenomena observed in nature.

Only a few attempts have been made to compare predictions of weak turbulence theory with numerical results. One can mention the results of Pushkarev and Zakharov [2] who numerically solved the three-dimensional equations for capillary water waves and observed a power-law spectrum close to that derived by Zakharov and Filonenko [3]. Majda, McLaughlin and Tabak [4] introduced a model, the so-called MMT model, for assessing numerically the validity of weak turbulence theory. Since their results indicated a failure of the predictions of weak turbulence theory, more computations have been carried out recently to get a better understanding of wave turbulence in the MMT model [5–7]. The present understanding is that co-

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herent structures can strongly affect weak turbulence. These coherent structures essentially are wave collapses and quasisolitons. Wave collapses in the form of sporadic localized events represent a strongly nonlinear mechanism of energy transfer towards small scales. Quasisolitons or envelope solitons denote approximate solutions of the MMT model which tend to classical solitons in the limit of a narrowbanded spectrum. The presence of such quasisolitons may explain the deviation from weak turbulence leading to the appearance of a steeper spectrum (the so-called MMT spectrum) in some cases [6,7]. Recently, Biven et al. [8] addressed the problem of breakdown of wave turbulence by intermittent events associated with nonlinear coherent structures.

In the present Letter we consider a model which is quite similar to the MMT model. However, coherent structures cannot develop. Our model takes the form of a system of equations describing the interactions of two types of waves. This is a fairly widespread case which includes, for example, the interaction of electrons with photons or the interaction of electromagnetic waves with Langmuir waves [1,9]. The main conclusion of this Letter is that numerical results based on the present model are in agreement with the predictions of weak turbulence theory. Agreement between numerical simulations and weak turbulence theory was also recently obtained by Zakharov et al. [10], who examined a modified version of the MMT model that allows for "one to three" wave interactions.

Of course, it will be necessary in the future to perform numerical computations on the full equations describing the physical phenomena of interest. However, we believe that a lot of information can be obtained from the solution of simplified models. Since the theory of weak turbulence is quite general, its main statements can be tested with simple models, for which numerical simulations can be performed more easily.

2. Model equations and Kolmogorov spectra

We consider the system of equations proposed in [7],

$$
\begin{aligned}
i\frac{\partial a_k}{\partial t} &= \omega_k a_k + \int T_{123k} a_1 b_2 b_3^* \\
&\quad \times \delta(k_1+k_2-k_3-k)\,dk_1\,dk_2\,dk_3, \\
i\frac{\partial b_k}{\partial t} &= s\omega_k b_k + \int T_{123k} b_1 a_2 a_3^* \\
&\quad \times \delta(k_1+k_2-k_3-k)\,dk_1\,dk_2\,dk_3,
\end{aligned} \tag{1}
$$

where a_k, b_k denote the Fourier components of two types of interacting wave fields and asterisk stands for complex conjugation. Like the MMT model, this model is determined by the linear dispersion relation $\omega_k = |k|^\alpha$ and the interaction coefficient $T_{123k} = |k_1 k_2 k_3 k|^{\beta/4}$. Thus ω_k, $s\omega_k$ and T_{123k} are homogeneous functions of their arguments. The three parameters s, α and β are real with the restriction $s, \alpha > 0$. If we set $\alpha = 2$ and $\beta = 0$, Eqs. (1) correspond to coupled nonlinear Schrödinger equations.

The system possesses two important conserved quantities, the positive definite Hamiltonian H, which we split into its linear part H_{L} and its nonlinear part H_{NL},

$$
\begin{aligned}
H &= H_{\mathrm{L}} + H_{\mathrm{NL}} \\
&= \int \omega_k \left(|a_k|^2 + s|b_k|^2\right) dk \\
&\quad + \int T_{123k} a_1 b_2 b_3^* a_k^* \delta(k_1+k_2-k_3-k) \\
&\quad \times dk_1\,dk_2\,dk_3\,dk,
\end{aligned}
$$

and the total wave action (or number of particles)

$$
N = \int \left(|a_k|^2 + |b_k|^2\right) dk.
$$

Note that both individual wave actions $\int |a_k|^2\,dk$ and $\int |b_k|^2\,dk$ are conserved in the system.

Eqs. (1) describe four-wave resonant interactions satisfying

$$
\begin{aligned}
&k_1 + k_2 = k_3 + k, \\
&\omega_1 + s\omega_2 = s\omega_3 + \omega_k.
\end{aligned} \tag{2}
$$

It is well known that when $s = 1$ conditions (2) have nontrivial solutions only if $\alpha < 1$. The case $s = 1$ and $\alpha = 1/2$, which mimics gravity waves in deep water, was treated in some recent studies [4–7]. In particular, Zakharov et al. [7] showed that the MMT model with $\alpha < 1$ exhibits coherent structures which strongly affect the weakly turbulent regime. Here accounting for $s \neq 1$ allows the resonance conditions (2) to be satisfied for any α. If $\alpha = 2$, we can solve explicitly Eqs. (2) to obtain

$$
k_3 = k_1 - \frac{2(k_1 - s k_2)}{1+s},
$$

$$k = k_2 + \frac{2(k_1 - sk_2)}{1+s}. \tag{3}$$

It is clear that Eqs. (3) with $s = 1$ give the trivial solution $k_3 = k_2$, $k = k_1$. As a general rule, for a given α, nontrivial families of resonant quartets obeying Eqs. (2) can be found for all values of $s \neq 1$.

In the framework of weak turbulence theory, we are interested in the evolution of the two-point correlation functions

$$\langle a_k a^*_{k'} \rangle = n^a_k \delta(k - k') \quad \text{and} \quad \langle b_k b^*_{k'} \rangle = n^b_k \delta(k - k'),$$

where $\langle\cdot\rangle$ represents ensemble averaging. Under the assumptions of random phases and quasi-Gaussianity, it is then possible to write a system of kinetic equations for n^a_k and n^b_k as

$$\frac{\partial n^a_k}{\partial t} = 2\pi \int |T_{123k}|^2 U^{ab}_{123k} \delta(\omega_1 + s\omega_2 - s\omega_3 - \omega_k) \times \delta(k_1 + k_2 - k_3 - k)\, dk_1\, dk_2\, dk_3, \tag{4}$$

$$\frac{\partial n^b_k}{\partial t} = 2\pi \int |T_{123k}|^2 U^{ba}_{123k} \delta(s\omega_1 + \omega_2 - \omega_3 - s\omega_k) \times \delta(k_1 + k_2 - k_3 - k)\, dk_1\, dk_2\, dk_3, \tag{5}$$

with

$$U^{ab}_{123k} = n^a_1 n^b_2 n^b_3 + n^a_1 n^b_2 n^a_k - n^a_1 n^b_3 n^a_k - n^b_2 n^b_3 n^a_k.$$

The stationary power-law solutions of Eqs. (4), (5) can be found explicitly. To do so, let us examine Eq. (4) only since the problem is similar for Eq. (5) by permuting n^a_k and n^b_k as well as ω_k and $s\omega_k$. Looking for solutions of the form $n^a_k \propto \omega_k^{-\gamma}$, $n^b_k \propto (s\omega_k)^{-\gamma}$ and applying Zakharov's conformal transformations, the kinetic equation (4) becomes

$$\frac{\partial \mathcal{N}^a_\omega}{\partial t} \propto \omega_k^{-y-1} I^a_{s\alpha\beta\gamma}, \tag{6}$$

where $\mathcal{N}^a_\omega = n^a_k\, dk/d\omega_k$ and

$$I^a_{s\alpha\beta\gamma} = \int_\Delta S_{123}\big[1 + (s\xi_3)^\gamma - (s\xi_2)^\gamma - \xi_1^\gamma\big] \times \delta(1 + s\xi_3 - s\xi_2 - \xi_1) \times \delta\big(1 + \xi_3^{1/\alpha} - \xi_2^{1/\alpha} - \xi_1^{1/\alpha}\big) \times \big[1 + (s\xi_3)^y - (s\xi_2)^y - \xi_1^y\big]\, d\xi_1\, d\xi_2\, d\xi_3, \tag{7}$$

with

$$\Delta = \{0 < \xi_1 < 1,\ 0 < s\xi_2 < 1,\ \xi_1 + s\xi_2 > 1\},$$

$$S_{123} = \frac{2\pi}{\alpha^4 s^{2\gamma}} (\xi_1\xi_2\xi_3)^{(\beta/2+1)/\alpha - 1 - \gamma},$$

and

$$y = 3\gamma + 1 - \frac{2\beta + 3}{\alpha}.$$

The nondimensionalized integral $I^a_{s\alpha\beta\gamma}$ in Eq. (6) results from the change of variables $\omega_j \to \omega_k \xi_j$ ($j = 1, 2, 3$).

Thermodynamic equilibrium solutions ($\gamma = 0, 1$) given by

$$n^{a,b}_k = \text{const} \quad \text{and} \quad n^{a,b}_k \propto \omega_k^{-1} \tag{8}$$

are obvious. In addition, there exist Kolmogorov-type solutions ($y = 0, 1$)

$$n^{a,b}_k \propto \omega_k^{(-2\beta/3 - 1 + \alpha/3)/\alpha} \quad \text{and} \quad n^{a,b}_k \propto \omega_k^{-(2\beta/3+1)/\alpha}, \tag{9}$$

which correspond to a finite flux of wave action Q and energy P, respectively. We point out that Eqs. (8), (9) are also steady solutions of Eq. (5) and they are identical to those derived from the MMT model [4]. The fact that the kinetic equation depends on the parameter s implies that the fluxes and the Kolmogorov constants also depend on s (see below). However, there is no s-dependence on the Kolmogorov exponents because of the property of scale invariance. As found in [7], the criterion for appearance of the Kolmogorov spectra (9) is

$$\beta < -\frac{3}{2} \quad \text{or} \quad \beta > 2\alpha - \frac{3}{2}. \tag{10}$$

This means physically that a flux of wave action towards large scales (inverse cascade with $Q < 0$) and a flux of energy towards small scales (direct cascade with $P > 0$) should occur in the system. The full expressions of Eq. (9) can be obtained from dimensional analysis yielding

$$n^a_k = c^a_1 Q_a^{1/3} \omega_k^{(-2\beta/3 - 1 + \alpha/3)/\alpha}, \quad n^b_k = c^b_1 Q_b^{1/3} (s\omega_k)^{(-2\beta/3 - 1 + \alpha/3)/\alpha}, \tag{11}$$

and

$$n^a_k = c^a_2 P_a^{1/3} \omega_k^{-(2\beta/3+1)/\alpha}, \quad n^b_k = c^b_2 P_b^{1/3} (s\omega_k)^{-(2\beta/3+1)/\alpha}, \tag{12}$$

where

$$c_1^{a,b} = \left(- \frac{\partial I_{s\alpha\beta\gamma}^{a,b}}{\partial y} \bigg|_{y=0} \right)^{-1/3},$$
$$c_2^{a,b} = \left(\frac{\partial I_{s\alpha\beta\gamma}^{a,b}}{\partial y} \bigg|_{y=1} \right)^{-1/3} \tag{13}$$

denote the dimensionless Kolmogorov constants. These can be computed directly by using integral (7) and its analogue for $\partial \mathcal{N}_\omega^b / \partial t$.

In the numerical computations, we will fix $\alpha = 3/2 > 1$ in order to prevent the emergence of coherent structures such as wave collapses and quasisolitons revealed in [7]. Our goal is to check the validity of the Kolmogorov spectra which are relevant in several real wave media as already said in the introduction [1]. We will restrict our study to solutions (12) associated with the direct cascade.

3. Numerical results

Numerical experiments were carried out to integrate Eqs. (1) by use of a pseudospectral code with periodic boundary conditions. The method includes a fourth-order Runge–Kutta scheme in combination with an integrating factor technique which permits efficient computations over long times [4,7]. Resolution with up to 2048 de-aliased modes in a domain of length 2π is achieved here ($k_{\max} = 1024$). To generate weakly turbulent regimes, source terms of the form

$$i \begin{pmatrix} f_k^a \\ f_k^b \end{pmatrix} e^{i\theta_k} - i \left[\begin{pmatrix} \nu_-^a \\ \nu_-^b \end{pmatrix} \left(k - k_d^-\right)^2 + \begin{pmatrix} \nu_+^a \\ \nu_+^b \end{pmatrix} \left(k - k_d^+\right)^2 \right] \times \begin{pmatrix} a_k \\ b_k \end{pmatrix} \tag{14}$$

were added to both right-hand sides of Eqs. (1). The first term in Eq. (14) denotes a white-noise forcing where $0 \leqslant \theta_k < 2\pi$ is an uniformly distributed random number varying in time. The term in square brackets consists of a wave action sink at large scales and an energy sink at small scales. The random feature of the forcing makes it uncorrelated in time with the wave field. Consequently, it is easier to control the input energy with a random forcing than with a deterministic forcing. For the results presented below, the forcing region is located at small wave numbers, i.e.,

$$f_k^{a,b} = \begin{cases} 6, 3 & \text{if } 8 \leqslant k \leqslant 12, \\ 0 & \text{otherwise.} \end{cases}$$

Parameters of the sinks are

$$\nu_-^{a,b} = \begin{cases} 16, 0.8 & \text{if } k \leqslant k_d^- \ (k_d^- = 5), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\nu_+^{a,b} = \begin{cases} 10^{-2}, 7 \times 10^{-4} & \text{if } k \geqslant k_d^+ \ (k_d^+ = 550), \\ 0 & \text{otherwise.} \end{cases}$$

Using this kind of selective dissipation ensures large enough inertial ranges at intermediate scales where solutions can develop under the negligible influence of damping. According to criterion (10), we focused on $\beta = 2$ and $s = 1/10$ as a typical case for testing weak turbulence predictions. Simulations are run from low-level initial data until a quasisteady state is reached and then averaging is performed over a sufficiently long time to compute the spectra. The time step, set equal to $\Delta t = 2 \times 10^{-5}$, has to resolve accurately the fastest harmonics $\tau \sim 1/\omega_{\max}$ of the medium or at least those from the inertial range. Time integration with such a small time step leads to a computationally time-consuming procedure despite the one-dimensionality of the problem. This explains why we chose $\alpha = 3/2$ rather than a greater integer value (e.g., $\alpha = 2$) as well as $s = 1/10$ rather than a value $s > 1$. Otherwise the constraint on Δt would be more severe. There is *a priori* no special requirement in the choice of the value of s, except $s \neq 1$.

Figs. 1 and 2 show the temporal evolution of the wave action N and the quadratic energy H_{L} over the window $80 \leqslant t \leqslant 100$. At this stage, the stationary regime is clearly established since the wave action and the quadratic energy fluctuate around some mean values $N \simeq 0.5$ and $H_{\mathrm{L}} \simeq 5.3$. Typically, the time interval for both the whole computation and the time averaging must exceed significantly the longest linear period. In order to monitor the level of turbulence, we define the average nonlinearity ϵ as the ratio of the nonlinear part to the linear part of the Hamiltonian, i.e.,

$$\epsilon = \frac{H_{\mathrm{NL}}}{H_{\mathrm{L}}}.$$

As in [2,7], this quantity provides a relatively good estimate of the level of nonlinearity once the system reaches the steady state. We can see in Fig. 3 that

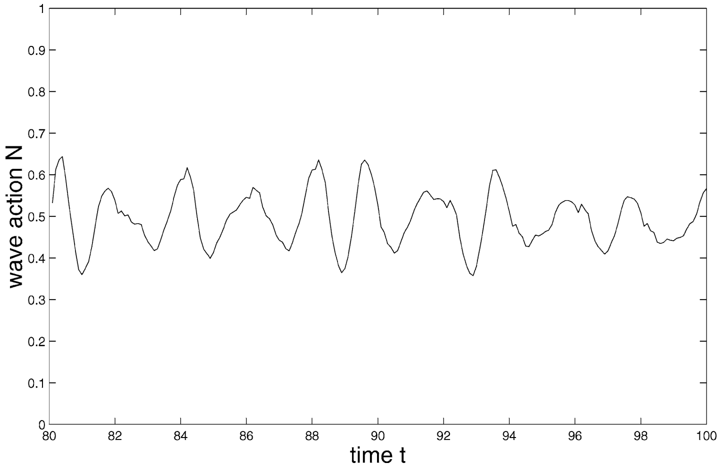

Fig. 1. $s = 1/10$, $\alpha = 3/2$, $\beta = 2$. Evolution of wave action N vs. time in the stationary state.

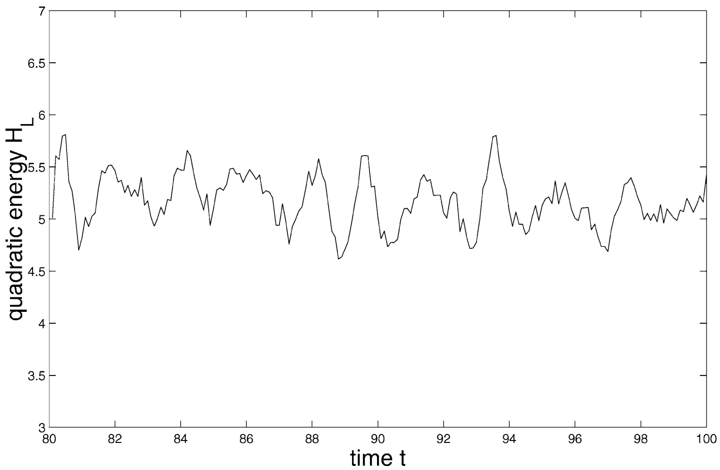

Fig. 2. $s = 1/10$, $\alpha = 3/2$, $\beta = 2$. Evolution of quadratic energy H_{L} vs. time in the stationary state.

the average nonlinearity fluctuates around some mean value $\epsilon \simeq 0.14$, which indicates that the condition of weak nonlinearity holds in our experiments. However, it should be emphasized that ϵ could not be imposed too small (by decreasing the forcing) otherwise the different modes would not be excited enough to generate an effective flux of energy. This problem is particularly important in numerics due to the discretization which restricts the possibilities for four-wave resonances. Since the effects of nonlinearity are assumed to be small in weak turbulence, it is sufficient to consider only the quantity H_{L} which contains the main part of the energy. We deduce the conservation of the total Hamiltonian from the conservation of H_{L} and ϵ, as illustrated in Figs. 2 and 3 because $H = H_{\mathrm{L}}(1+\epsilon)$.

Fig. 4 displays the stationary isotropic spectra $n_k^{a,b}$ realized in the present situation. By comparison, we also plotted the predicted Kolmogorov solutions given by Eq. (12). For $\alpha = 3/2$ and $\beta = 2$, they read

$$n_k^a = c_2^a P_a^{1/3} \omega_k^{-14/9} = c_2^a P_a^{1/3} k^{-7/3}, \tag{15}$$

and

$$n_k^b = c_2^b P_b^{1/3} (s\omega_k)^{-14/9} = c_2^b P_b^{1/3} s^{-14/9} k^{-7/3}, \quad s = 1/10, \tag{16}$$

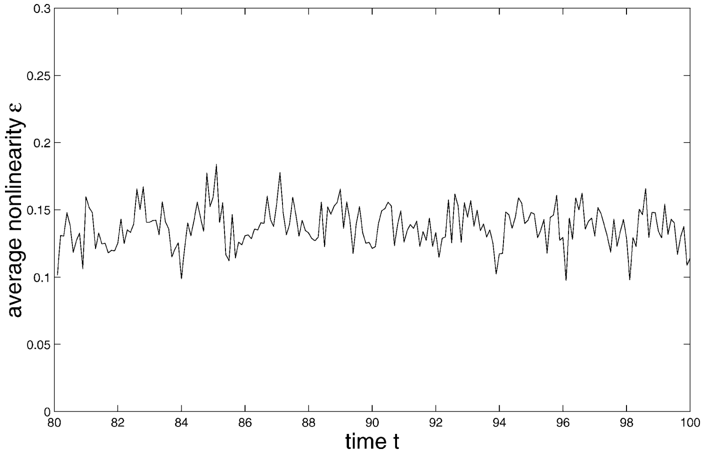

Fig. 3. $s = 1/10$, $\alpha = 3/2$, $\beta = 2$. Evolution of average nonlinearity ϵ vs. time in the stationary state.

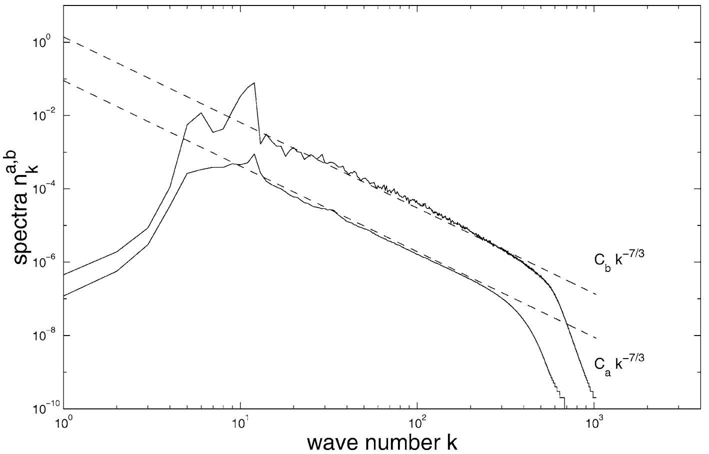

Fig. 4. $s = 1/10$, $\alpha = 3/2$, $\beta = 2$. Computed spectra (n_k^a for the lower one and n_k^b for the upper one) and predicted Kolmogorov spectra $C_{a,b}k^{-7/3}$ with $C_a = c_2^a P_a^{1/3}$ and $C_b = c_2^b P_b^{1/3} s^{-14/9}$ (dashed lines).

where $c_2^a = 0.094$ and $c_2^b = 0.047$ are numerically calculated from Eq. (13). The mean fluxes of energy $P_{a,b}$ in Eqs. (15) and (16) can be expressed as

$$P_a = 2 \int\limits_{k > k_d^+} \nu_+^a \left(k - k_d^+\right)^2 \omega_k n_k^a \, dk$$

and

$$P_b = 2 \int\limits_{k > k_d^+} \nu_+^b \left(k - k_d^+\right)^2 s \omega_k n_k^b \, dk,$$

with k_d^+ the cutoff of ultraviolet dissipation [1,7]. Then it is straightforward to get their values $P_a = 0.86$ and $P_b = 0.56$ from simulations. We can observe in Fig. 4 that for both wave fields the spectra are well approximated by the Kolmogorov power-laws over a wide range of scales (say $20 < k < 300$). Here the agreement between theory and numerics is found with respect to both the slope and the level of the spectra.

4. Conclusion

We have studied a simplified one-dimensional model describing media with two types of interacting waves. The regime of parameters has been chosen such that weak turbulence theory can be applied correctly and coherent structures cannot develop. This way we avoid any interference between weak wave turbulence and coherent structures. Our numerical results show the appearance of a pure weak turbulence state with the formation of a complete Kolmogorov spectrum. This suggests the general relevance of weak turbulence even in one-dimensional systems. In the future it will be of interest to extend the present work to higher dimensions by still considering simplified models such as the present one, and to perform computations on the full equations describing the physical phenomena of interest. Recall that Pushkarev and Zakharov [2] successfully observed weak turbulence for capillary water waves in three dimensions. However, their numerical simulations based on the truncated basic equations were very time-consuming and the Kolmogorov spectrum that they measured extended only over a small range of scales.

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