Chapter 13: Complex Numbers
Sections 13.1 & 13.2
1. Complex numbers

- Complex numbers are of the form

\[ z = x + iy, \quad x, y \in \mathbb{R}, \quad i^2 = -1. \]

- In the above definition, \( x \) is the real part of \( z \) and \( y \) is the imaginary part of \( z \).

- The complex number \( z = x + iy \) may be represented in the complex plane as the point with cartesian coordinates \((x, y)\).
The complex conjugate of $z = x + iy$ is defined as

$$\bar{z} = x - iy.$$

As a consequence of the above definition, we have

$$\Re(z) = \frac{z + \bar{z}}{2}, \quad \Im(z) = \frac{z - \bar{z}}{2i}, \quad z\bar{z} = x^2 + y^2. \quad (1)$$

If $z_1$ and $z_2$ are two complex numbers, then

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \quad z_1z_2 = \overline{z_1} \overline{z_2}. \quad (2)$$
The absolute value or modulus of $z = x + iy$ is

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$ 

It is a positive number.

**Examples:** Evaluate the following
- $|i|$
- $|2 - 3i|$
2. Algebra of complex numbers

- You should use the same rules of algebra as for real numbers, but remember that $i^2 = -1$.

**Examples:**

- # 13.1.1: Find powers of $i$ and $1/i$.
- Assume $z_1 = 2 + 3i$ and $z_2 = -1 - 7i$. Calculate $z_1 z_2$ and $(z_1 + z_2)^2$.

- Get used to writing a complex number in the form

$$z = (\text{real part}) + i (\text{imaginary part}),$$

no matter how complicated this expression might be.
Remember that multiplying a complex number by its complex conjugate gives a real number.

**Examples:** Assume $z_1 = 2 + 3i$ and $z_2 = -1 - 7i$.

- Find $\frac{z_1}{z_2}$.
- Find $\frac{\overline{z}_1}{\overline{z}_2}$.
- Find $\Im \left( \frac{1}{z^3} \right)$.
- **# 13.2.27:** Solve $z^2 - (8 - 5i)z + 40 - 20i = 0$. 


3. Polar coordinates form of complex numbers

- In polar coordinates,
  \[ x = r \cos(\theta), \quad y = r \sin(\theta), \]
  where
  \[ r = \sqrt{x^2 + y^2} = |z|. \]

- The angle \( \theta \) is called the argument of \( z \). It is defined for all \( z \neq 0 \), and is given by
  \[
  \arg(z) = \theta = \begin{cases} 
  \arctan \left( \frac{y}{x} \right) & \text{if } x \geq 0 \\
  \arctan \left( \frac{y}{x} \right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\
  \arctan \left( \frac{y}{x} \right) - \pi & \text{if } x < 0 \text{ and } y < 0
  \end{cases} \pm 2n\pi.
  \]
Principal value \( \text{Arg}(z) \)

- Because \( \text{arg}(z) \) is multi-valued, it is convenient to agree on a particular choice of \( \text{arg}(z) \), in order to have a single-valued function.

- The principal value of \( \text{arg}(z) \), \( \text{Arg}(z) \), is such that

\[
\tan(\text{Arg}(z)) = \frac{y}{x}, \quad \text{with } -\pi < \text{Arg}(z) \leq \pi.
\]

- Note that \( \text{Arg}(z) \) jumps by \(-2\pi\) when one crosses the negative real axis from above.
Principal value $\text{Arg}(z)$ (continued)

- **Examples:**
  - Find the principal value of the argument of $z = 1 - i$.
  - Find the principal value of the argument of $z = -10$. 

![Complex number plane with axes labeled x and y.](image)
Polar and cartesian forms of a complex number

- You need to be able to go back and forth between the polar and cartesian representations of a complex number.

\[ z = x + iy = |z| \cos(\theta) + i|z| \sin(\theta). \]

- In particular, you need to know the values of the sine and cosine of multiples of \( \pi/6 \) and \( \pi/4 \).
  - Convert \( \cos \left( \frac{\pi}{6} \right) + i \sin \left( \frac{\pi}{6} \right) \) to cartesian coordinates.
  - What is the cartesian form of the complex number such that \( |z| = 3 \) and \( \text{Arg}(z) = \pi/4 \)?
Euler’s formula

- Euler’s formula reads
  \[ \exp(i\theta) = \cos(\theta) + i\sin(\theta), \quad \theta \in \mathbb{R}. \]

- As a consequence, every complex number \( z \neq 0 \) can be written as
  \[ z = |z| (\cos(\theta) + i\sin(\theta)) = |z| \exp(i\theta). \]

- This formula is extremely useful for calculating powers and roots of complex numbers, or for multiplying and dividing complex numbers in polar form.
Integer powers of a complex number

To find the $n$-th power of a complex number $z \neq 0$, proceed as follows

1. Write $z$ in exponential form,
   
   $$z = |z| \exp (i\theta).$$

2. Then take the $n$-th power of each side of the above equation
   
   $$z^n = |z|^n \exp (in\theta) = |z|^n (\cos(n\theta) + i \sin(n\theta)).$$

3. In particular, if $z$ is on the unit circle ($|z| = 1$), we have
   
   $$(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta).$$

   This is De Moivre’s formula.
Examples of application:

- Trigonometric formulas

\[
\begin{aligned}
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta), \\
\sin(2\theta) &= 2\sin(\theta)\cos(\theta).
\end{aligned}
\]  

Find \(\cos(3\theta)\) and \(\sin(3\theta)\) in terms of \(\cos(\theta)\) and \(\sin(\theta)\).
Product of two complex numbers

- The **product** of \( z_1 = r_1 \exp (i \theta_1) \) and \( z_2 = r_2 \exp (i \theta_2) \) is

\[
    z_1 z_2 = (r_1 \exp (i \theta_1)) (r_2 \exp (i \theta_2)) = r_1 r_2 \exp (i (\theta_1 + \theta_2)).
\]  

(4)

- As a consequence,

\[
    \arg(z_1 z_2) = \arg(z_1) + \arg(z_2), \quad |z_1 z_2| = |z_1| |z_2|.
\]

- We can use Equation (4) to show that

\[
    \cos (\theta_1 + \theta_2) = \cos (\theta_1) \cos (\theta_2) - \sin (\theta_1) \sin (\theta_2),
\]

\[
    \sin (\theta_1 + \theta_2) = \sin (\theta_1) \cos (\theta_2) + \cos (\theta_1) \sin (\theta_2).
\]

(5)
Similarly, the ratio \( \frac{z_1}{z_2} \) is given by

\[
\frac{z_1}{z_2} = \frac{r_1 \exp(i \theta_1)}{r_2 \exp(i \theta_2)} = \frac{r_1}{r_2} \exp(i(\theta_1 - \theta_2)).
\]

As a consequence,

\[
\arg \left( \frac{z_1}{z_2} \right) = \arg(z_1) - \arg(z_2), \quad \left| \frac{z_1}{z_2} \right| = \left| \frac{z_1}{z_2} \right|. 
\]

**Example:** Assume \( z_1 = 2 + 3i \) and \( z_2 = -1 - 7i \). Find \( \left| \frac{z_1}{z_2} \right| \).
To find the $n$-th roots of a complex number $z \neq 0$, proceed as follows

1. Write $z$ in exponential form,

$$z = r \exp\left(i(\theta + 2p\pi)\right),$$

with $r = |z|$ and $p \in \mathbb{Z}$.

2. Then take the $n$-th root (or the $1/n$-th power)

$$\sqrt[n]{z} = z^{1/n} = r^{1/n} \exp\left(i \frac{\theta + 2p\pi}{n}\right) = \sqrt[n]{r} \exp\left(i \frac{\theta + 2p\pi}{n}\right).$$

3. There are thus $n$ roots of $z$, given by

$$z_k = \sqrt[n]{r} \left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right)\right), \quad k = 0, \ldots, n-1.$$
Roots of a complex number (continued)

- The principal value of $\sqrt[n]{z}$ is the $n$-th root of $z$ obtained by taking $\theta = \text{Arg}(z)$ and $k = 0$.
- The $n$-th roots of unity are given by
  \[
  \sqrt[n]{1} = \cos \left( \frac{2k\pi}{n} \right) + i \sin \left( \frac{2k\pi}{n} \right) = \omega^k, \quad k = 0, \ldots, n - 1
  \]
  where $\omega = \cos(2\pi/n) + i \sin(2\pi/n)$.
- In particular, if $w_1$ is any $n$-th root of $z \neq 0$, then the $n$-th roots of $z$ are given by
  \[
  w_1, \ w_1\omega, \ w_1\omega^2, \ldots, \ w_1\omega^{n-1}.
  \]
Examples:

- Find the three cubic roots of 1.

- Find the four values of \( 4\sqrt{i} \).

- Give a representation in the complex plane of the principal value of the eighth root of \( z = -3 + 4i \).
Triangle inequality

- If $z_1$ and $z_2$ are two complex numbers, then
  
  $$|z_1 + z_2| \leq |z_1| + |z_2|.$$  

  This is called the **triangle inequality**. Geometrically, it says that the length of any side of a triangle cannot be larger than the sum of the lengths of the other two sides.

- More generally, if $z_1, z_2, \ldots, z_n$ are $n$ complex numbers, then
  
  $$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$