Chapter 13: Complex Numbers
Sections 13.5, 13.6 & 13.7

1. Complex exponential

- The exponential of a complex number \( z = x + iy \) is defined as

  \[
  \exp(z) = \exp(x + iy) = \exp(x) \exp(iy) = \exp(x) (\cos(y) + i \sin(y)).
  \]

- As for real numbers, the exponential function is equal to its derivative, i.e.

  \[
  \frac{d}{dz} \exp(z) = \exp(z). \tag{1}
  \]

- The exponential is therefore entire.

- You may also use the notation \( \exp(z) = e^z \).
Properties of the exponential function

- The exponential function is periodic with period $2\pi i$: indeed, for any integer $k \in \mathbb{Z}$,
  \[ \exp(z + 2k\pi i) = \exp(x)(\cos(y + 2k\pi) + i\sin(y + 2k\pi)) = \exp(x)(\cos(y) + i\sin(y)) = \exp(z). \]
- Moreover,
  \[ |\exp(z)| = |\exp(x)| |\exp(iy)| = \exp(x) \sqrt{(\cos^2(y) + \sin^2(y))} = \exp(x) = \exp(\Re(e(z))). \]
- As with real numbers,
  - $\exp(z_1 + z_2) = \exp(z_1) \exp(z_2)$;
  - $\exp(z) \neq 0$.

2. Trigonometric functions

- The complex sine and cosine functions are defined in a way similar to their real counterparts,
  \[ \cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}. \]
- The tangent, cotangent, secant and cosecant are defined as usual. For instance,
  \[ \tan(z) = \frac{\sin(z)}{\cos(z)}, \quad \sec(z) = \frac{1}{\cos(z)}, \quad \text{etc.} \]
The rules of differentiation that you are familiar with still work.

**Example:**
- Use the definitions of \( \cos(z) \) and \( \sin(z) \),

\[
\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.
\]

to find \((\cos(z))'\) and \((\sin(z))'\).

- Show that Euler’s formula also works if \( \theta \) is complex.

### 3. Hyperbolic functions

- The complex hyperbolic sine and cosine are defined in a way similar to their real counterparts,

\[
\cosh(z) = \frac{e^z + e^{-z}}{2}, \quad \sinh(z) = \frac{e^z - e^{-z}}{2}.
\]  

(3)

- The hyperbolic sine and cosine, as well as the sine and cosine, are **entire**.

- We have the following relations

\[
\cosh(iz) = \cos(z), \quad \sinh(iz) = i \sin(z),
\]

\[
\cos(iz) = \cosh(z), \quad \sin(iz) = i \sinh(z).
\]  

(4)
4. Complex logarithm

- The logarithm $w$ of $z \neq 0$ is defined as $e^w = z$.

- Since the exponential is $2\pi i$-periodic, the complex logarithm is multi-valued.

- Solving the above equation for $w = w_r + iw_i$ and $z = re^{i\theta}$ gives

$$e^w = e^{w_r}e^{iw_i} = re^{i\theta} \implies \begin{cases} e^{w_r} = r \\ w_i = \theta + 2p\pi \end{cases},$$

which implies $w_r = \ln(r)$ and $w_i = \theta + 2p\pi$, $p \in \mathbb{Z}$.

- Therefore,

$$\ln(z) = \ln(|z|) + i \arg(z).$$

Principal value of $\ln(z)$

- We define the principal value of $\ln(z)$, $\text{Ln}(z)$, as the value of $\ln(z)$ obtained with the principal value of $\arg(z)$, i.e.

$$\text{Ln}(z) = \ln(|z|) + i \text{Arg}(z).$$

- Note that $\text{Ln}(z)$ jumps by $-2\pi i$ when one crosses the negative real axis from above.

- The negative real axis is called a branch cut of $\text{Ln}(z)$. 
Principal value of \( \ln(z) \) (continued)

- Recall that
  \[
  \ln(z) = \ln(|z|) + i \text{Arg}(z).
  \]
- Since \( \text{Arg}(z) = \arg(z) + 2p\pi \), \( p \in \mathbb{Z} \), we therefore see that \( \ln(z) \) is related to \( \ln(z) \) by
  \[
  \ln(z) = \ln(z) + i 2p\pi, \quad p \in \mathbb{Z}.
  \]

**Examples:**
- \( \ln(2) = \ln(2) \), but \( \ln(2) = \ln(2) + i 2p\pi, \quad p \in \mathbb{Z} \).
- Find \( \ln(-4) \) and \( \ln(-4) \).
- Find \( \ln(10i) \).

Properties of the logarithm

- You have to be careful when you use identities like
  \[
  \ln(z_1 z_2) = \ln(z_1) + \ln(z_2), \quad \text{or} \quad \ln \left( \frac{z_1}{z_2} \right) = \ln(z_1) - \ln(z_2).
  \]
  They are only true up to multiples of \( 2\pi i \).
- For instance, if \( z_1 = i = \exp(i\pi/2) \) and \( z_2 = -1 = \exp(i\pi) \),
  \[
  \ln(z_1) = i \frac{\pi}{2} + 2p_1i\pi, \quad \ln(z_2) = i\pi + 2p_2i\pi, \quad p_1, p_2 \in \mathbb{Z},
  \]
  and
  \[
  \ln(z_1 z_2) = i \frac{3\pi}{2} + 2p_3i\pi, \quad p_3 \in \mathbb{Z},
  \]
  but \( p_3 \) is not necessarily equal to \( p_1 + p_2 \).
Properties of the logarithm (continued)

Moreover, with \( z_1 = i = \exp(i\pi/2) \) and \( z_2 = -1 = \exp(i\pi) \),

\[
\text{Ln}(z_1) = i \frac{\pi}{2}, \quad \text{Ln}(z_2) = i \pi,
\]

and

\[
\text{Ln}(z_1 z_2) = -i \frac{\pi}{2} \neq \text{Ln}(z_1) + \text{Ln}(z_2).
\]

However, every branch of the logarithm (i.e. each expression of \( \text{Ln}(z) \) with a given value of \( p \in \mathbb{Z} \)) is analytic except at the branch point \( z = 0 \) and on the branch cut of \( \text{Ln}(z) \). In the domain of analyticity of \( \text{Ln}(z) \),

\[
\frac{d}{dz} (\text{Ln}(z)) = \frac{1}{z}. \quad (5)
\]

5. Complex power function

If \( z \neq 0 \) and \( c \) are complex numbers, we define

\[
z^c = \exp(c \text{Ln}(z)) = \exp(c \text{Ln}(z) + 2pc\pi i), \quad p \in \mathbb{Z}.
\]

For \( c \in \mathbb{C} \), this is again a multi-valued function, and we define the principal value of \( z^c \) as

\[
z^c = \exp(c \text{Ln}(z))
\]