1. Matrices and vectors

- An \( m \times n \) matrix is an array with \( m \) rows and \( n \) columns. It is typically written in the form

\[
A = [a_{ij}] = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix},
\]

where \( i \) is the row index and \( j \) is the column index.

- A column vector is an \( m \times 1 \) matrix. Similarly, a row vector is a \( 1 \times n \) matrix.

- The entries \( a_{ij} \) of a matrix \( A \) may be real or complex.
Matrices and vectors (continued)

- **Examples:**
  - $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a $2 \times 2$ square matrix with **real** entries.
  
  - $u = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a column vector of $A$.
  
  - $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 3-7i \end{bmatrix}$ is a $3 \times 3$ diagonal matrix, with complex entries.
  
  - An $n \times n$ diagonal matrix whose entries are all ones is called the $n \times n$ identity matrix.
  
  - $C = \begin{bmatrix} 1 & 2 & 3 & 10 \\ 1 & 6 & -8 & 0 \end{bmatrix}$ is a $2 \times 4$ matrix with **real** entries.

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Matrix addition and scalar multiplication

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices, and let $c$ be a scalar.

- The matrices $A$ and $B$ are **equal** if and only if they have the same entries,
  $A = B \iff a_{ij} = b_{ij}, \text{ for all } i, j, \ 1 \leq i \leq m, \ 1 \leq j \leq n.$

- The **sum** of $A$ and $B$ is the $m \times n$ matrix obtained by adding the entries of $A$ to those of $B$,
  $A + B = [a_{ij} + b_{ij}].$

- The **product** of $A$ with the scalar $c$ is the $m \times n$ matrix obtained by multiplying the entries of $A$ by $c$,
  $cA = [c \cdot a_{ij}].$
2. Matrix multiplication

Let \( A = [a_{ij}] \) be an \( m \times n \) matrix and \( B = [b_{ij}] \) be an \( n \times p \) matrix. The product \( C = AB \) of \( A \) and \( B \) is an \( m \times p \) matrix whose entries are obtained by multiplying each row of \( A \) with each column of \( B \) as follows:

\[
c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.
\]

**Examples:** Let \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \) and \( C = \begin{bmatrix} 1 & 2 & 3 & 10 \\ 1 & 6 & -8 & 0 \end{bmatrix} \).

- Is the product \( AC \) defined? If so, evaluate it.
- Same question with the product \( CA \).
- What is the product of \( A \) with the third column vector of \( C \)?

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Matrix multiplication (continued)

**More examples:**

- Consider the system of equations

\[
\begin{align*}
3x_1 + 2x_2 - x_3 &= 4 \\
x_2 - 7x_3 &= 0 \\
-x_1 + 4x_2 - 6x_3 &= -10
\end{align*}
\]

Write this system in the form \( AX = Y \), where \( A \) is a matrix and \( X \) and \( Y \) are two column vectors.

- Let

\[
A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.
\]

Calculate the products \( AB \) and \( BA \).
3. Rules for matrix addition and multiplication

- The rules for matrix addition and multiplication by a scalar are the same as the rules for addition and multiplication of real or complex numbers.

- In particular, if $A$ and $B$ are matrices and $c_1$ and $c_2$ are scalars, then

\[
A + B = B + A
\]
\[
(A + B) + C = A + (B + C)
\]
\[
c_1 (A + B) = c_1 A + c_1 B
\]
\[
(c_1 + c_2)A = c_1A + c_2A
\]
\[
c_1 (c_2 A) = (c_1 c_2)A
\]

whenever the above quantities make sense.

- The product of two matrices is associative and distributive, i.e.

\[
A(BC) = (AB)C = ABC
\]
\[
A(B + C) = AB + AC
\]
\[
(A + B)C = AC + BC.
\]

- However, the product of two matrices is not commutative. If $A$ and $B$ are two square matrices, we typically have

\[
AB \neq BA
\]

- For two square matrices $A$ and $B$, the commutator of $A$ and $B$ is defined as

\[
[A, B] = AB - BA.
\]

In general, $[A, B] \neq 0$. If $[A, B] = 0$, one says that the matrices $A$ and $B$ commute.
4. Transposition

- The transpose of an $m \times n$ matrix $A$ is the $n \times m$ matrix $A^T$ obtained from $A$ by switching its rows and columns, i.e.

$$A = [a_{ij}] \quad \text{then} \quad A^T = [a_{ji}].$$

- **Example:** Find the transpose of $C = \begin{bmatrix} 1 & 2 & 3 & 10 \\ 1 & 6 & -8 & 0 \end{bmatrix}$.

- **Some properties of transposition.** If $A$ and $B$ are matrices, and $c$ is a scalar, then

$$ (A + B)^T = A^T + B^T \quad (cA)^T = cA^T \quad (AB)^T = B^T A^T \quad \left(A^T\right)^T = A,$$

whenever the above quantities make sense.

5. Linear independence

- A **linear combination** of the $n$ vectors $a_1, a_2, \ldots, a_n$ is an expression of the form

$$c_1 a_1 + c_2 a_2 + \cdots + c_n a_n,$$

where the $c_i$'s are scalars.

- A set of vectors $\{a_1, a_2, \ldots, a_n\}$ is **linearly independent** if the only way of having a linear combination of these vectors equal to zero is by choosing all of the coefficients equal to zero. In other words, $\{a_1, a_2, \ldots, a_n\}$ is linearly independent if and only if

$$c_1 a_1 + c_2 a_2 + \cdots + c_n a_n = 0 \implies c_1 = c_2 = \cdots = c_n = 0.$$
6. Vector space

A real (or complex) vector space is a non-empty set \( V \) whose elements are called vectors, and which is equipped with two operations called vector addition and multiplication by a scalar.

The vector addition satisfies the following properties.

1. The sum of two vectors \( a \in V \) and \( b \in V \) is denoted by \( a + b \) and is an element of \( V \).
2. It is commutative: \( a + b = b + a \), for all \( a, b \in V \).
3. It is associative: \((a + b) + c = a + (b + c)\) for all \( a, b, c \in V \).
4. There exists a unique zero vector, denoted by 0, such that for every vector \( a \in V \), \( a + 0 = a \).
5. For each \( a \in V \), there exists a unique vector \((-a) \in V \) such that \( a + (-a) = 0 \).
Vector space (continued)

- The **multiplication by a scalar** satisfies the following properties.
  1. The multiplication of a vector \( a \in V \) by a scalar \( \alpha \in \mathbb{R} \) (or \( \alpha \in \mathbb{C} \)) is denoted by \( \alpha a \) and is an element of \( V \).
  2. Multiplication by a scalar is **distributive**:
     \[
     \alpha (a + b) = \alpha a + \alpha b, \quad (\alpha + \beta) a = \alpha a + \beta a,
     \]
     for all \( a, b \in V \) and \( \alpha, \beta \in \mathbb{R} \) (or \( \mathbb{C} \)).
  3. It is **associative**: \( \alpha (\beta a) = (\alpha \beta) a \) for all \( a \in V \) and \( \alpha, \beta \in \mathbb{R} \) (or \( \mathbb{C} \)).
  4. Multiplying a vector by 1 gives back that vector, i.e.
     \[
     1 a = a,
     \]
     for all \( a \in V \).

Bases and dimension

- The **span** of set of vectors \( U = \{a_1, a_2, \cdots, a_n\} \) is the set of all linear combinations of vectors in \( U \). It is denoted by
  \[
  \text{Span}\{a_1, a_2, \cdots, a_n\} \text{ or Span}(U)
  \]
  and is a **subspace** of \( V \).
- A **basis** \( B \) of a subspace \( S \) of \( V \) is a set of vectors of \( S \) such that
  1. \( \text{Span}(B) = S \);
  2. \( B \) is a linearly independent set.
- **Theorem**: If a basis \( B \) of a subspace \( S \) of \( V \) has \( n \) vectors, then all other bases of \( S \) have exactly \( n \) vectors.
- The **dimension** of a vector space \( V \) (or of a subspace \( S \) of \( V \)) spanned by a finite number of vectors is the number of vectors in any of its bases.
7. Rank

- The **row space** of an \( m \times n \) matrix \( A \) is the span of the row vectors of \( A \). If \( A \) has real entries, the row space of \( A \) is a subspace of \( \mathbb{R}^n \).

- Similarly, the **column space** of \( A \) is the span of the column vectors of \( A \), and is a subspace of \( \mathbb{R}^m \).

- The **rank** of a matrix \( A \) is the dimension of its column space.

- **Theorem**: The dimensions of the row and column spaces of a matrix \( A \) are the same. They are equal to the rank of \( A \).

- **Example**: Check that the row and column spaces of
  \[
  C = \begin{bmatrix}
  1 & 2 & 3 & 10 \\
  1 & 6 & -8 & 0
  \end{bmatrix}
  \]
  are vector subspaces, and find their dimension.

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The rank theorem

- The **null space** of an \( m \times n \) matrix \( A \), \( \mathcal{N}(A) \) is the set of vectors \( u \) such that \( Au = 0 \). If \( A \) has real entries, then \( \mathcal{N}(A) \) is a subspace of \( \mathbb{R}^n \).

- The **rank theorem** states that if \( A \) is an \( m \times n \) matrix, then

  \[
  \text{rank}(A) + \dim(\mathcal{N}(A)) = n.
  \]

- **Example**: Find the rank and the null space of the matrix
  \[
  C = \begin{bmatrix}
  1 & 2 & 3 & 10 \\
  1 & 6 & -8 & 0
  \end{bmatrix}
  \]
  Check that the rank theorem applies.