Chapters 7-8: Linear Algebra
Sections 7.5, 7.8 & 8.1
1. Linear systems of equations

- A **linear system** of equations of the form
  
  \[
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
  \vdots \\
  a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
  \]

  can be written in matrix form as \( AX = B \), where

  \[
  A = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
  \end{bmatrix}, \quad
  X = \begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
  \end{bmatrix}, \quad
  B = \begin{bmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_m
  \end{bmatrix}
  \]
Solution(s) of a linear system of equations

Given a matrix $A$ and a vector $B$, a solution of the system $AX = B$ is a vector $X$ which satisfies the equation $AX = B$.

If $B$ is not in the column space of $A$, then the system $AX = B$ has no solution. One says that the system is not consistent. In the statements below, we assume that the system $AX = B$ is consistent.

If the null space of $A$ is non-trivial, then the system $AX = B$ has more than one solution.

The system $AX = B$ has a unique solution provided $\dim(\mathcal{N}(A)) = 0$.

Since, by the rank theorem, $\text{rank}(A) + \dim(\mathcal{N}(A)) = n$ (recall that $n$ is the number of columns of $A$), the system $AX = B$ has a unique solution if and only if $\text{rank}(A) = n$. 
A linear system of the form \( AX = 0 \) is said to be homogeneous.

Solutions of \( AX = 0 \) are vectors in the null space of \( A \).

If we know one solution \( X_0 \) to \( AX = B \), then all solutions to \( AX = B \) are of the form

\[
X = X_0 + X_h
\]

where \( X_h \) is a solution to the associated homogeneous equation \( AX = 0 \).

In other words, the general solution to the linear system \( AX = B \), if it exists, can be written as the sum of a particular solution \( X_0 \) to this system, plus the general solution of the associated homogeneous system.
2. Inverse of a matrix

- If $A$ is a square $n \times n$ matrix, its inverse, if it exists, is the matrix, denoted by $A^{-1}$, such that
  \[ AA^{-1} = A^{-1} A = I_n, \]
  where $I_n$ is the $n \times n$ identity matrix.

- A square matrix $A$ is said to be singular if its inverse does not exist. Similarly, we say that $A$ is non-singular or invertible if $A$ has an inverse.

- The inverse of a square matrix $A = [a_{ij}]$ is given by
  \[ A^{-1} = \frac{1}{\det(A)} [C_{ij}]^T, \]
  where $\det(A)$ is the determinant of $A$ and $C_{ij}$ is the matrix of cofactors of $A$.  

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The determinant of a square $n \times n$ matrix $A = [a_{ij}]$ is the scalar

$$\text{det}(A) = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

where the cofactor $C_{ij}$ is given by

$$C_{ij} = (-1)^{i+j} M_{ij},$$

and the minor $M_{ij}$ is the determinant of the matrix obtained from $A$ by “deleting” the $i$-th row and $j$-th column of $A$.

Example: Calculate the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. 
Properties of determinants

- If a determinant has a row or a column entirely made of zeros, then the determinant is equal to zero.

- The value of a determinant does not change if one replaces one row (resp. column) by itself plus a linear combination of other rows (resp. columns).

- If one interchanges 2 columns in a determinant, then the value of the determinant is multiplied by $-1$.

- If one multiplies a row (or a column) by a constant $C$, then the determinant is multiplied by $C$.

- If $A$ is a square matrix, then $A$ and $A^T$ have the same determinant.
Properties of the inverse

- Since the inverse of a square matrix $A$ is given by
  \[ A^{-1} = \frac{1}{\det(A)} [C_{ij}]^T, \]
  we see that $A$ is invertible if and only if $\det(A) \neq 0$.

- If $A$ is an invertible $2 \times 2$ matrix, 
  \[
  \begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
  \end{bmatrix},
  \]
  then
  \[ A^{-1} = \frac{1}{\det(A)} \begin{bmatrix}
  a_{22} & -a_{12} \\
  -a_{21} & a_{11}
  \end{bmatrix}, \]
  and $\det(A) = a_{11}a_{22} - a_{21}a_{12}$.

- If $A$ and $B$ are invertible, then
  \[(AB)^{-1} = B^{-1}A^{-1} \quad \text{and} \quad (A^{-1})^{-1} = A.\]
Consider the following linear system of $n$ equations with $n$ unknowns,

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
  \vdots \\
  a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{align*}
\]

This system can be also be written in matrix form as $AX = B$, where $A$ is a square matrix.

If $\det(A) \neq 0$, then the above system has a unique solution $X$ given by

\[X = A^{-1}B.\]
Linear systems of equations - summary

Consider the linear system $AX = B$ where $A$ is an $m \times n$ matrix.

- The system may not be consistent, in which case it has no solution.

- To decide whether the system is consistent, check that $B$ is in the column space of $A$.

- If the system is consistent, then
  - Either $\text{rank}(A) = n$ (which also means that $\text{dim}(\mathcal{N}(A)) = 0$), and the system has a unique solution.
  - Or $\text{rank}(A) < n$ (which also means that $\mathcal{N}(A)$ is non-trivial), and the system has an infinite number of solutions.
Consider the linear system $AX = B$ where $A$ is an $m \times n$ matrix.

- If $m = n$ and the system is consistent, then
  - Either $\det(A) \neq 0$, in which case $\text{rank}(A) = n$, $\dim(\mathcal{N}(A)) = 0$, and the system has a unique solution;
  - Or $\det(A) = 0$, in which case $\dim(\mathcal{N}(A)) > 0$, $\text{rank}(A) < n$, and the system has an infinite number of solutions.

- Note that when $m = n$, having $\det(A) = 0$ means that the columns of $A$ are linearly dependent.
- It also means that $\mathcal{N}(A)$ is non-trivial and that $\text{rank}(A) < n$. 
Let $A$ be a square $n \times n$ matrix. We say that $X$ is an eigenvector of $A$ with eigenvalue $\lambda$ if

$$X \neq 0 \quad \text{and} \quad AX = \lambda X.$$ 

The above equation can be re-written as

$$(A - \lambda I_n)X = 0.$$

Since $X \neq 0$, this implies that $A - \lambda I_n$ is not invertible, i.e. that $\det(A - \lambda I_n) = 0$.

The eigenvalues of $A$ are therefore found by solving the characteristic equation $\det(A - \lambda I_n) = 0$. 

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The characteristic polynomial \( \det(A - \lambda I_n) \) is a polynomial of degree \( n \) in \( \lambda \). It has \( n \) complex roots, which are not necessarily distinct from one another.

If \( \lambda \) is a root of order \( k \) of the characteristic polynomial \( \det(A - \lambda I_n) \), we say that \( \lambda \) is an eigenvalue of \( A \) of algebraic multiplicity \( k \).

If \( A \) has real entries, then its characteristic polynomial has real coefficients. As a consequence, if \( \lambda \) is an eigenvalue of \( A \), so is \( \bar{\lambda} \).

If \( A \) is a \( 2 \times 2 \) matrix, then its characteristic polynomial is of the form \( \lambda^2 - \lambda \text{Tr}(A) + \det(A) \), where the trace of \( A \), \( \text{Tr}(A) \), is the sum of the diagonal entries of \( A \).
Examples: Find the eigenvalues of the following matrices.

- \( A = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \).
- \( B = \begin{bmatrix} -1 & 9 \\ 0 & 5 \end{bmatrix} \).
- \( C = \begin{bmatrix} -13 & -36 \\ 6 & 17 \end{bmatrix} \).
- \( D = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ -1 & 1 & 2 \end{bmatrix} \).
Once an eigenvalue $\lambda$ of $A$ has been found, one can find an associated eigenvector, by solving the linear system

$$(A - \lambda I_n) X = 0.$$ 

Since $\mathcal{N}(A - \lambda I_n)$ is not trivial, there is an infinite number of solutions to the above equation. In particular, if $X$ is an eigenvector of $A$ with eigenvalue $\lambda$, so is $\alpha X$, where $\alpha \in \mathbb{R}$ (or $\mathbb{C}$) and $\alpha \neq 0$.

The set of eigenvectors of $A$ with eigenvalue $\lambda$, together with the zero vector, form a subspace of $\mathbb{R}^n$ (or $\mathbb{C}^n$), $E_\lambda$, called the eigenspace of $A$ corresponding to the eigenvalue $\lambda$.

The dimension of $E_\lambda$ is called the geometric multiplicity of $\lambda$. 

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Examples: Find the eigenvectors of the following matrices. Each time, give the algebraic and geometric multiplicities of the corresponding eigenvalues.

- $A = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$.
- $C = \begin{bmatrix} -13 & -36 \\ 6 & 17 \end{bmatrix}$.
- $D = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ -1 & 1 & 2 \end{bmatrix}$. 
The geometric multiplicity $m_\lambda$ of an eigenvalue $\lambda$ is less than or equal to its algebraic multiplicity $M_\lambda$.

If $M_\lambda = 1$, then $m_\lambda = 1$.

If $m_\lambda$ is not equal to $M_\lambda$, then one can find $M_\lambda - m_\lambda$ linearly independent generalized eigenvectors of $A$, by solving a sequence of equations of the form

$$ (A - \lambda I_n) U_{i+1} = U_i, \quad i \in \{1, \ldots, M_\lambda - m_\lambda\} $$

where $U_1 = X_\lambda$ is a genuine eigenvector of $A$ with eigenvalue $\lambda$. 

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Examples: Find the genuine and generalized eigenvectors of the following matrices

\[ M = \begin{bmatrix}
4 & 1 & 0 & 0 \\
0 & 4 & 0 & 0 \\
0 & 0 & 4 & 1 \\
0 & 0 & 0 & 4
\end{bmatrix}. \]

\[ N = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}. \]

If \( A \) has \( k \) distinct eigenvalues and \( \mathcal{B}_1, \cdots, \mathcal{B}_k \) are bases of the corresponding generalized eigenspaces, then \( \{\mathcal{B}_1, \cdots, \mathcal{B}_k\} \) is a basis of \( \mathbb{R}^n \) (or \( \mathbb{C}^n \)).