1A. Suppose $A$ is an $n \times n$ matrix over a field $F$ and that $A^k = 0$ for some $k \in \mathbb{N}$. Show that $A^n = 0$.

1B. Let $V$ be a vector space over a field $K$. Show that the following two statements are equivalent:
   
   (a) we have $\dim_K(V) < \infty$;
   
   (b) any $K$-linear endomorphism $f : V \to V$ satisfies an equation of the form $f^m + a_{m-1}f^{m-1} + \cdots + a_1f + a_0$, where $m \in \mathbb{N}$ and $a_0, \ldots, a_{m-1} \in K$.

2A. Let $G$ be a finite group of order $2^n(2m + 1)$. Let $H := \text{Hom}(G, \mathbb{Z}/2\mathbb{Z})$. Show that $H$ is an abelian group whose order is $2^s$ for some number $s \in \{0, 1, \ldots, n\}$. Give examples when $s = 0$ and when $s = n$.

2B. If $G$ is a group of order $4n + 2$ show that $G$ has a subgroup $H$ of index $2$.

3A. If $R$ is a ring show that all non-zero-divisors in $R$ have the same additive order. What are the possible orders?

3B. Let $K$ be a field. Let $R = K[x, y]$ be the polynomial ring in two variables with coefficients in $K$. For $a \in K$, let $(x^2, y + ax)$ be the ideal of $R$ generated by $x^2$ and $y + ax$. We consider the ideal $I := \cap_{a \in K}(x^2, y + ax)$ of $R$. Compute the smallest number $n \in \mathbb{N}$ such that $I$ is generated by $n$ elements. Give an example of $n$ elements $i_1, \ldots, i_n \in I$ that generate $I$.

4A. Let $K$ be a Galois extension of $\mathbb{Q}$ of degree 4. Let $D$ be the set of those square free integers $d \in \mathbb{Z} \setminus \{1\}$ which have the property that $\mathbb{Q}(\sqrt{d}) \subseteq K$.

   (a) Show that the set $D$ has either 1 or 3 elements.
(b) If $D = \{d_1, d_2, d_3\}$ has three elements, then show that the product $d_1d_2d_3$ is a perfect square.

(c) If $K = \mathbb{Q}(\zeta_5)$, find $D$.

4B. Determine a Galois closure $K$ for $L = \mathbb{Q}(3i - \sqrt{2})$ over $\mathbb{Q}$, and determine the Galois group $G = G(K: \mathbb{Q})$.

5A. An abelian group has generators $a$, $b$, $c$, $d$ and defining relations $3b + 2c + 8d = 0$, $5a + b - 4c + 8d = 0$, $-2a + b + 4c - 8d = 0$, and $-a + 3b + 2c + 8d = 0$. Express the group as a direct sum of cyclic groups.

5B. Let $K = \mathbb{F}_5 = \{[0], [1], [2], [3], [4]\}$. Let $V := K^3$. We consider the four vectors $v_1 = ([2], [3], [1])$, $v_2 = ([1], [2], [4])$, $v_3 = ([0], [1], [1])$, and $v_4 = ([4], [2], [1])$. Show that $B = \{v_1, v_2, v_3\}$ is a $K$-basis of $V$. Compute the coordinates of $v_4$ with respect to $B$. 