1. Let \( a_n \) for \( n = 0, 1, 2, 3, \ldots \) be a sequence of real numbers. Consider the power series
\[
\sum_{n=0}^{\infty} a_n x^n.
\]
a. Prove that there exists \( R \) with \( 0 < R < \infty \) such that the series converges absolutely for all real \( x \) with \( |x| < R \) and the series diverges for all real \( x \) with \( |x| > R \).
b. Find the radius of convergence \( R \) for the series \( \sum_{n=0}^{\infty} n^3 a_n x^n \).

2. The real number line \( \mathbb{R} \) with the usual metric \( d(x, y) = |x - y| \) is a complete metric space. Consider the set \( X = (0, +\infty) \) of strictly positive real numbers. Denote the usual metric by \( d_1(x, y) = |x - y| \). Consider also the unusual metric
\[
d_2(x, y) = |x - y| + \left| \frac{1}{x} - \frac{1}{y} \right|.
\]
a. Are the two metric spaces \( (X, d_1) \) and \( (X, d_2) \) topologically isomorphic (homeomorphic)? Answer yes or no, and prove that your answer is correct.
b. Is \( (X, d_1) \) a complete metric space? Answer yes or no, and prove that your answer is correct.
c. Is \( (X, d_2) \) a complete metric space? Answer yes or no, and prove that your answer is correct.

3. In each case, state whether the assertion is true or false, and then prove or disprove it. In the first two parts the context is Borel measurable real functions on the closed unit interval \([0, 1]\). The measure is Lebesgue measure.
a. \( L^1([0, 1]) \subset L^2([0, 1]) \).
b. Suppose for each \( n = 1, 2, 3, \ldots \) and for each \( x \) in \([0, 1]\) the function \( f_n(x) \geq 0 \), that \( \int_0^1 f_n(x) \, dx \leq 1 \), and that for each \( x \) in \([0, 1]\) the sequence \( f_n(x) \to 0 \) as \( n \to \infty \). Then it follows that \( \int_0^1 f_n(x) \, dx \to 0 \) as \( n \to \infty \). Then it follows that \( \int_0^1 f_n(x) \, dx \to 0 \) as \( n \to \infty \).
c. \( \ell^1 \subset \ell^2 \).
4. The Fourier transform of a function \( f \) in \( L^1 \) (with respect to Lebesgue measure on the real line) is the function \( \hat{f} \) defined for real \( k \) by

\[
\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) \, dx.
\]

a. Prove that for \( f \) and \( g \) each in \( L^1 \) the products \( fg \) and \( f\hat{g} \) are also in \( L^1 \).

b. Prove that for \( f \) and \( g \) each in \( L^1 \) that

\[
\int_{-\infty}^{\infty} \hat{f}(k)g(k) \, dk = \int_{-\infty}^{\infty} f(x)\hat{g}(x) \, dx.
\]

5. a. Prove that for each \( t > 0 \) the function \( g(y) = (1 + t/y)^y \) is strictly increasing for \( y > 0 \).

b. Let \( a > 0 \). Find

\[
\lim_{n \to \infty} \int_0^1 \left( 1 - a\frac{\log(x)}{n} \right)^n dx.
\]

(The logarithm is base \( e \).) Prove that your calculation is correct. If you use a limit theorem, specify it and prove that the hypotheses are satisfied.

6. In this problem the measurable functions are Borel measurable real functions on the closed interval \([-1,1]\). Let \( \lambda \) denote Lebesgue measure on this interval, and let \( \mu = \lambda + \delta \), where \( \delta \) is a unit point mass Dirac measure at the origin.

a. Show that \( \lambda \) is absolutely continuous with respect to \( \mu \).

A measurable function \( h \geq 0 \) is a Radon-Nikodym derivative of \( \lambda \) with respect to \( \mu \) if \( \lambda(f) = \mu(hf) \) for all bounded measurable functions \( f \). (Here \( \lambda \) and \( \mu \) denote integration with respect to the corresponding measures.) Two such Radon-Nikodym derivatives are equal except on a set of \( \mu \) measure zero.

b. Describe explicitly the Radon-Nikodym derivative \( h \) of \( \lambda \) with respect to \( \mu \).

The space \( L^\infty \) consists of equivalence classes of essentially bounded measurable functions with respect to the appropriate measure, two functions being identified if they are equal almost everywhere with respect to the measure. If two measurable functions are equal almost everywhere with respect to \( \mu \), then they are equal almost everywhere with respect to \( \lambda \). Thus there is a natural map \( T \) from \( L^\infty(\mu) \) to \( L^\infty(\lambda) \).

c. Is \( T \) injective? Prove or disprove.

d. Is \( T \) surjective? Prove or disprove.