Please choose one problem (A or B) of each of the following six pairs of problems. Please show all your work; in particular, explain clearly all steps (such as taking a limit under the integral sign or changing the order of integration) by quoting known theorems and, when appropriate, by verifying that their assumptions are satisfied. GOOD LUCK!

1A. Calculate
\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T \sin \sqrt{2} t \, dt. \]

1B. Given a sequence \((a_n)\), \(n = 0, 1, 2, \ldots\), let
\[ s_n = a_0 + a_1 + \ldots + a_n \]
be its \(n\)-th partial sum. If the limit
\[ A = \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^N s_n \]
exists, we say that the sequence \((a_n)\) is Cesàro-summable to \(A\). Find the Cesàro sum \(A\) of the sequence \(e^{inz}\) where \(z \in \mathbb{R}\).

2A. Let \(X\) be the set of all real sequences
\[ x = (\alpha_1, \alpha_2, \ldots) \]
such that
\[ \|x\| := \sup_n \left| \sum_{j=1}^n \alpha_j \right| < +\infty. \]
Show that \(X\) is a linear space and that \(\| \cdot \|\) is a norm on \(X\). Does \(\sum_1^\infty |\alpha_n| < +\infty\) imply that \(x \in X\)? Does \(x \in X\) imply that \(\sum_1^\infty |\alpha_n| < +\infty\)?
2B. Let $M_n(C)$ denote the linear space of complex $n \times n$ matrices. For $A \in M_n(C)$, let $A^*$ denote the conjugate transpose of $A$, i.e., for each $i$ and $j$ the elements of matrices $A$ and $A^*$ are related by:

$$(A^*)_{ij} = A_{ji}.$$ 

Prove that

$$d(A, B) = (\text{Tr}[(A - B)(A^* - B^*)])^{\frac{1}{2}}$$

defines a metric on $M_n(C)$. Here $\text{Tr}$ denotes the trace of a matrix.

3A. Let $f$ be the function

$$f(x) = \begin{cases} \frac{2}{x^2}, & \text{if } x \geq 1, \\ \exp x, & \text{if } x < 1. \end{cases}$$

Show that its cosine transform,

$$\hat{f}(k) := \int_{\mathbb{R}} \cos(kx) f(x) \, dx,$$

is a continuous function of $k$.

3B. Let $f \in L^2([0, 2\pi])$ with $\int_0^{2\pi} |f(\theta)|^2 \, d\theta = 1$ and let

$$f_n := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(\theta) \exp(-in\theta) \, d\theta$$

be the Fourier coefficients of $f$, with respect to the orthonormal basis $\frac{1}{\sqrt{2\pi}} \exp(in\theta)$, $n \in \mathbb{Z}$.

Show: for every integer $k > 0$, at most $k^2$ of the $f_n$ can satisfy $|f_n| \geq \frac{1}{k}$.

4A. Let $f \in L^\infty([0, 1])$ (with Lebesgue measure). Prove that the function

$$g(p) = \|f\|_p = \left(\int_0^1 |f(x)|^p \, dx\right)^{\frac{1}{p}}$$

is a nondecreasing function of $p \in [1, \infty]$. 


4B Let \( f_n : \mathbb{R} \to \mathbb{R}, n = 1, 2, 3, \ldots \) be Lebesgue-measurable functions. Suppose there exists a constant \( C \) such that
\[
\int_{\mathbb{R}} |f_n| \leq \frac{C}{n^{3/4}}
\]
for all \( n \). Prove that \( f_n \to 0 \) Lebesgue-almost everywhere.

5A. Calculate the integral
\[
\int_{|x+y|\leq 1} \ln |x+y| \, dx \, dy.
\]

5B. Calculate the integral
\[
\int_{\mathbb{R}^n} e^{-[x_1^2+(x_2+x_3)^2+\cdots+(x_1+x_2+\cdots+x_n)^2]} \, dx_1 \, dx_2 \ldots \, dx_n.
\]

In problems 6A and 6B we use the following definition: a sequence \( (\mu_n) \) of probability measures on \([0,1]\) converges weakly to a measure \( \mu \) if
\[
\int_0^1 f \, d\mu_n \to \int_0^1 f \, d\mu
\]
as \( n \to \infty \) for every continuous function \( f : [0,1] \to \mathbb{R} \). (Recall that for a probability measure \( \nu \), \( \int_0^1 \, d\nu = 1 \).)

6A. For \( a \in [0,1] \), let \( \delta_a \) be the measure defined by:
\[
\delta_a(E) = \begin{cases} 
1 & \text{if } a \in E; \\
0 & \text{if } a \notin E.
\end{cases}
\]
\( \delta_a \) is called the point mass at \( a \). In the following formula for clearer notation we write \( \delta_{\frac{x^2}{n^2}} \) instead of \( \delta_{\frac{x^2}{n^2}} \). Let
\[
\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\frac{i^2}{n^2}}.
\]
Find the weak limit of the measures \( \mu_n \).

6B. Let the measures \( \mu_n, n = 1, 2, \ldots \) and \( \mu \) on \([0,1]\) satisfy
\[
\int_0^1 t^k \, \mu_n(dt) \to \int_0^1 t^k \, \mu(dt)
\]
as \( n \to \infty \) for \( k = 0, 1, 2, \ldots \). Prove that \( \mu_n \) converges weakly to \( \mu \).