Geometry—Topology Qualifying Exam
Fall 2000

1. Find all meromorphic functions on the Riemann sphere with a simple pole at \( z = 1 \), a simple zero at \( z = 0 \) and no other poles or zeros. Justify your answer.

2. Suppose \( X \) and \( Y \) are metric spaces and \( f : X \rightarrow Y \) is a map from \( X \) to \( Y \). Show that \( f \) is continuous if and only if \( f \) maps convergent sequences in \( X \) into convergent sequences in \( Y \).

3. Let \( M = \{(x, y)| y^2 - x^2 = 1, y > 0\} \subset \mathbb{R}^2 \). Then \( M \) is a one dimensional manifold with the global coordinate function \( x \). For \( a \in \mathbb{R} \) define

\[
R_a \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cosh a & \sinh a \\ \sinh a & \cosh a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cosh a \cdot x + \sinh a \cdot y \\ \sinh a \cdot x + \cosh a \cdot x \end{bmatrix}.
\]

Then it is easy to check that \( R_a : M \rightarrow M \) and also \( R_a R_b = R_{a+b} \) (note: \( \cosh a = \frac{e^a + e^{-a}}{2} \) and \( \sinh a = \frac{e^a - e^{-a}}{2} \)).

Let \( p_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) be the point in \( M \) with \( x \) coordinate equal to 0. Every point, \( p \), in \( M \) can be expressed in the form \( p = R_a p_0 \) for a unique choice of \( a \in \mathbb{R} \). For \( p = R_a p_0 \) define a vector field \( V(p) = dR_{a,0} \left( \frac{\partial}{\partial x} \right)_{x=0} \) where \( dR_{a,0} : T_{p_0}(M) \rightarrow T_p(M) \) is the derivative of the map \( R_a \) at \( p_0 \) and \( \left( \frac{\partial}{\partial x} \right)_{x=0} \) is the vector field associated with the coordinate function \( x \) evaluated at \( p_0 \). Show that \( V(R_b p) = dR_b V(p) \) and find the expression \( v(x) \frac{\partial}{\partial x} \) for the vector field \( V \) in the coordinate system \( x \).

4. The Laplacian, \( \Delta \), on \( \mathbb{R}^n \) is given by

\[
\Delta f = \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2}.
\]

An important property of the Laplacian is that it is symmetric on \( C_0^\infty(\mathbb{R}^n) \). That is,

\[
\int_{\mathbb{R}^n} (\Delta f(x)g(x) - f(x)\Delta g(x)) \, dx_1 \wedge \cdots \wedge dx_n = 0
\]

for \( f, g \in C_0^\infty \). Prove this by showing that the \( n \)-form

\[
(\Delta f(x)g(x) - f(x)\Delta g(x)) \, dx_1 \wedge \cdots \wedge dx_n;
\]

is exact for \( f, g \in C_\infty \). Then apply an appropriate version of Stokes’ theorem.

5. Let \( X \) denote the torus with 2 circles pinched to points (see the picture below). Use the Mayer-Vietoris sequence to compute the singular homology \( H_*(X, \mathbb{Z}) \).
6. Let \( p : \mathbb{R} \to S^1 \) be defined by \( p(t) = e^{2\pi i t} = \cos 2\pi t + i \sin 2\pi t \). Note that \( p \) is a covering map. Prove or give a counter example to the following statement: If \( f : \mathbb{R}P^2 \to S^1 \) is continuous then there exists a continuous lift \( \tilde{f} : \mathbb{R}P^2 \to \mathbb{R} \) so that \( f = p \circ \tilde{f} \).

7. Let \( X \) be the set of \( 2 \times 2 \) upper triangular complex matrices with determinant 1. Note that \( X \) is a 4 dimensional differentiable manifold. For which integers \( i \) do there exist closed \( i \) forms on \( X \) which are not exact?