1. For each of the items below, give an example, or briefly explain why none exists:
   (a) a nonconstant holomorphic function \( f : \mathbb{C} \to \mathbb{C} \) which is periodic with respect to \( \mathbb{Z} \), i.e. \( f(z + n) = f(z) \) for all \( z \in \mathbb{C} \) and \( n \in \mathbb{Z} \);
   (b) a meromorphic function \( g \) on \( \mathbb{C} \) having Taylor series centered at 0 with radius of convergence exactly 3;
   (c) a meromorphic function \( h \) on \( \mathbb{C} \), without poles on \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \), such that \( \int_{S^1} f(z)dz \neq 0 \) (where \( S^1 \) has the counterclockwise orientation).

[Note: A meromorphic function in \( \mathbb{C} \) is holomorphic except at a discrete set of points, and for each of these exceptional points, the function has at most a pole type singularity].

2. Consider the function
   \[ F : \mathbb{R}^4 \to \mathbb{R}^2 : \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} t \\ -t^2 + x^2 + y^2 + tz^2 \end{pmatrix} \]
   (a) Determine the sets of critical points and critical values of \( F \).
   (b) Because the domain of \( F \) is not compact, the topology of the level sets \( F^{-1}(u,v) \) can change as \( \begin{pmatrix} u \\ v \end{pmatrix} \) varies in a connected component of the set of regular values. Verify this in this example by determining the homotopy type of the level sets corresponding to regular values of the form \( \begin{pmatrix} u \\ 1 \end{pmatrix} \).

3. Consider the surface \( X \) in \( \mathbb{R}^3 \) which is defined in cylindrical coordinates by the equation
   \[ (r - 2)^2 + z^2 = 1. \]
   [Recall that in cylindrical coordinates \( x = r\cos(\theta), y = r\sin(\theta) \), and \( z = z \)].
   (a) Find an explicit basis for the homology (in all degrees) of this surface.
   (b) By introducing coordinates or parameterizing the surface, find explicit expressions for forms which represent the corresponding dual basis in DeRham cohomology.
4. For each $t \in \mathbb{R}$, let $\phi_t$ denote the map of $S^2$ into itself which is defined by

$$
\phi_t : S^2 \to S^2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \cos(t)x - \sin(t)y \\ \sin(t)x + \cos(t)y \\ z \end{pmatrix}.
$$

Show that $\phi_t$ is a one-parameter group of diffeomorphisms, and compute and graph the vector field $\mathbf{v}$ on $S^2$ for which $\phi_t$ is the corresponding flow.

Remark: In defining the vector field $\mathbf{v}$, please specify how you are viewing the tangent bundle of $S^2$.

5. (a) Let $\mathbb{C}^*$ denote the nonzero complex numbers. Determine all the covering spaces (up to isomorphism) of $\mathbb{C}^*$ and their automorphism groups.

(b) Let $G$ denote a finite subgroup of $SU(2, \mathbb{C})$, the group of unit quaternions (which you may identify with $SU(2, \mathbb{C})$, if you prefer). Using covering space theory, explain why $G$ is isomorphic to the fundamental group of the coset space $\mathbb{H}_1 / G$.

6. Let $X$ denote a manifold, let $c_2 : I^2 \to X$ denote a singular two-cube, and let $\eta$ denote a 1-form on $X$. Stokes' Theorem implies that

$$
\int_{c_2} d\eta = \int_{\partial c_2} \eta.
$$

Explain the meaning of this statement, i.e. the definitions of the left and right hand sides, and explain, as briefly as possible, why this equality ultimately follows from the fundamental theorem of calculus.

7. Consider the 1-form $\alpha = xdy - ydx$ in $\mathbb{R}^3$. Prove the following: if $f(x, y, z) \in C^\infty(\mathbb{R}^3)$ and $f \alpha$ is a closed 1-form, then $f$ is identically zero.

[Hint: use cylindrical coordinates].