Geometry-Topology Qualifying Exam

Do just 1 of the first 2 problems:

1. Compute the following integral,
   \[ \int_0^\infty \frac{\sqrt{x}}{1 + x^2} \, dx. \]

2. Find the image of the vertical strip \(|Re| < \frac{3}{2}\) (\(Re\) is the real part of \(z\)) under the conformal map \(z \rightarrow \sin z\).

3. Let \(G\) be the group of transformations of \(\mathbb{R}^2\) generated by the maps \(\alpha(x, y) = (x, y + 1)\) and \(\beta(x, y) = (x + 1, -y)\):
   a) Show that \(G\) acts evenly on \(\mathbb{R}^2\) (recall that \(G\) acts evenly if every point in \(\mathbb{R}^2\) has a neighborhood \(V\) so that \(gV \cap hV = \emptyset\) if \(g, h \in G\) and \(g \neq h\)). “Identify” the quotient \(\mathbb{R}^2/G\) by determining the identifications induced by \(G\) on the boundary of the square \(|x| \leq \frac{1}{2}\) and \(|y| \leq \frac{1}{2}\).
   b) You may take it as known that \(\mathbb{R}^2 \rightarrow \mathbb{R}^2/G\) is the universal covering of \(\mathbb{R}^2/G\) and for \(p \in \mathbb{R}^2/G\), the fundamental group \(\pi_1(\mathbb{R}^2/G, p)\) is isomorphic to \(G\). Show that the group, \(H\), generated by \(\alpha\) and \(\beta^2\) is a normal subgroup of \(\pi_1(\mathbb{R}^2/G, p)\) and determine the covering space \(X\) of \(\mathbb{R}^2/G\) associated with this subgroup of the fundamental group. Determine the group of deck transformations of the covering space \(X \rightarrow \mathbb{R}^2/G\).

4. The connected sum, \(X = \mathbb{RP}^2 \# \mathbb{RP}^2\), of two copies of real projective 2-space, \(\mathbb{RP}^2\), can be pictured as the union of two (relatively) open sets \(U\) and \(V\), pictured below, which intersect in the shaded annular region.

   ![Diagram of \(\mathbb{RP}^2 \# \mathbb{RP}^2\)]

   a) Use the Seifert-van Kampen theorem to determine the fundamental group of \(X\). Use this to also compute the first homology group \(H_1(X)\).
   b) Use Mayer-Vietoris to calculate the deRham cohomology \(H^k(X)\) for \(k = 0, 1, 2\). Explain any results you require to evaluate the terms in the Mayer-Vietoris sequence.

5. Let \(M\) denote the submanifold of \(\mathbb{R}^3\) defined by \(x^2 - z^2 - y^2 = 1\) with \(z > 0\). Define \(\varphi_t(x, y, z) = (sz + cx, y, cz + sz)\) where \(c = \cosh t\) and \(s = \sinh t\).
   a) Show that \(t \rightarrow \varphi_t\) determines a one-parameter family of diffeomorphisms of \(M\) and find the vector field associated with \(t \rightarrow \varphi_t\) in the coordinates for \(M\) obtained by projection \((x, y, z) \rightarrow (x, y)\).
   b) Determine \(\varphi_t^* \left( \frac{dx \wedge dy}{z} \right)\).

6. Use the implicit function theorem for smooth maps defined on open domains in Euclidean space to show that for a differentiable map \(f : M \rightarrow N\) and an embedded submanifold \(W \subset N\), the inverse image \(f^{-1}(W)\), will be an embedded submanifold of \(M\) whenever each point of \(W\) is a regular value of \(f\).

7. Let \(M\) be a connected manifold which is the union of two open submanifolds \(M = A \cup B\). Assume the \(A\) and \(B\) are orientable and \(A \cap B\) is connected. Show that \(M = A \cup B\) is orientable.