The Leaking Bucket Problem

A bucket of water of mass 20 kg is pulled at constant velocity up to a platform 40 meters above the ground. This takes ten minutes during which time 5 kg of water drips out at a steady rate through a hole in the bottom of the bucket. Find the work needed to raise the bucket to the platform.

Solution: The bucket moves upward at \( \frac{40}{10} = 4 \) meters/minute. If time is in minutes the bucket is at a height of \( x = 4t \) meters above the ground at time \( t \). The water drips out at a rate of \( \frac{5}{10} \) kg/minute. Since there is initially 20 kg of water in the bucket, at time \( t \) there is

\[
m = 20 - 0.5t \text{ kg}
\]

remaining. Consider the time interval between \( t \) and \( t + \Delta t \). During this time the bucket moves a distance \( \Delta x = 4 \Delta t \) meters. So the work done during this interval is

\[
mg \Delta x = (20 - 0.5t)g4 \Delta t
\]

and the integral for the total work done is

\[
4g \int_{0}^{10} (20 - 0.5 \cdot t) dt
\]

\[
= 4g(20t - 0.25t^2)\bigg|_{0}^{10} = 700g = 700 \cdot (9.8) = 6860 \text{ joules.}
\]
Alternate Solution (Colin Tisdale, student 051 Section 25):
Let $x$ be the height from the ground where the bucket starts, so the division is $\Delta x$. Since $W = F \cdot d$ and using the fact that $x = 4t$ or $t = -x/4$, we get

\[
\text{Mass} = 20 - \frac{t}{2},
\]
\[
\text{Mass} = 20 - \frac{x}{8}
\]

and this mass is moved only $\Delta x$ so

\[
\Delta W = (20 - \frac{x}{8}) \cdot g \Delta x
\]

so $
W = g \int_0^{40} 20 - \frac{x}{8} dx$

\[
g\left(20x - \frac{x^2}{16}\right)_{0}^{40} = 79.8(800 - \frac{1600}{16})
\]

\[
= 9.8(700) = 6860 \text{ Joules}.
\]
as before. Well done!

An alternate form of the integral is:

\[
\int_0^{18} (196 - 1.225y) dy
\]
Alternate Solution: (Jason Hooley, student 051 Section 25): We will calculate the work required to lift the bucket intact, less the work not done because of the leak. We will have to use the acceleration due to gravity because this is a metric units problem. The work for the intact bucket is \(20(40)9.8 = 7840\) joules. For the work not done,

\[
\begin{align*}
\Delta m &= 5 - \frac{t}{2} \\
\Delta F &= (5 - \frac{t}{2})9.8 \\
\Delta W &= (5 - \frac{t}{2})9.8\Delta y \\
&= (5 - \frac{t}{2})9.8(4\Delta t) \\
\text{Lost Work} &= 39.2 \int_0^{10} \left(5 - \frac{t}{2}\right) dt \\
&= 39.2 \left[5t - \frac{t^2}{2}\right]_0^{10} \\
&= 39.2(50 - 25) \\
&= 39.2 \cdot 25 \\
&= 980
\end{align*}
\]

Thus the net work done is \(7840 - 980 = 6860\), as before.