CALCULATION OF SECOND PARTIALS;
The Gory Details

Using the $UV$ table, the independent variables are Year ($Y$) and Degrees of Latitude ($L$), and the dependent variable is Ultraviolet Radiation Units ($UV$). The point of fascination is $(2000, -70)$, where there are interesting numbers.

I. To get

$$\frac{\partial^2 UV}{\partial Y^2}$$

we need to realize that what is changing is

$$\frac{\partial UV}{\partial Y}$$

so calculate

$$\frac{\partial UV}{\partial Y}$$

twice to get the change in the change.

A. Go left:

$$\frac{\partial UV}{\partial Y} \approx \frac{\Delta UV}{\Delta Y} = \frac{10.36 - 6.36}{2000 - 1995} = \frac{4}{5} = .8 \frac{UV}{Y}$$

B. Go right

$$\frac{\partial UV}{\partial Y} \approx \frac{\Delta UV}{\Delta Y} = \frac{11.46 - 10.36}{2005 - 2000} = \frac{1.1}{5} = .22 \frac{UV}{Y}$$

C. Thus we have:

- An estimate of the partial of $UV$ with respect to $Y$ from 1995 to 2000 of .8, so assign this estimate to 1997.5, the midpoint
- An estimate of the partial of $UV$ with respect to $Y$ from 2000 to 2005 of .22, which we will assign to 2002.5, the midpoint

D. So

$$\left( \frac{\Delta \frac{\partial UV}{\partial Y}}{\Delta Y} \right) = \frac{.22 - .8}{5} = -\frac{.116 \frac{UV}{Y}}{5} \approx \frac{\partial^2 UV}{\partial Y^2}$$
that is the rate of change of $\frac{\partial UV}{\partial t}$ measured in $\frac{UV}{yr}$ is -.116 such $\frac{UV}{yr}$ per year, which we abbreviate $\frac{UV}{yr}$ just like feet-per-second-squared from physics class.

II. Thundering onward

$$\frac{\partial^2 UV}{\partial L^2}$$

is the same, only as Tweedledee said, “contrariwise”

A Go up, 7.64 is for -60 which is larger than -70 because the smaller number is the larger number with minus signs, right?

$$\frac{\partial UV}{\partial L} \approx \frac{\Delta UV}{\Delta L} = \frac{7.64 - 10.36}{-60 - -70} = \frac{-2.72}{10} = -0.272 \frac{UV}{L}$$

B. Go down

$$\frac{\partial UV}{\partial L} \approx \frac{\Delta UV}{\Delta L} = \frac{10.36 - 6.71}{-70 - -80} = +0.365 \frac{UV}{L}$$

C. And with feeling:

$$\left( \frac{\Delta \frac{\partial UV}{\partial L}}{\Delta L} \right) = \frac{-0.272 - 0.365}{10} = \frac{-0.637}{10} = -0.0637 \frac{UV}{L} \approx \frac{\partial^2 UV}{\partial L^2}$$

using the same midpoint trick.

III. La pièce de résistance – the crosspartials – the first being:

$$\frac{\partial^2 UV}{\partial L \partial Y}$$

which is

$$\frac{\partial}{\partial L} \left( \frac{\partial UV}{\partial Y} \right)$$

is it not? Thus the thing which is changing (whose rate of change it is our solemn duty to find) is

$$\frac{\partial UV}{\partial Y}$$

and I need to find how it changes when $L$ changes.
So looking in the table, using bigger \( L \)-numbers, which are really smaller numbers because of the minus signs I see:
\[
\begin{align*}
1.41 & \quad 7.64 & \quad 10.82 & \quad \text{at } L = -60^\circ
\end{align*}
\]
Thus

A. Up, which is high:
\[
\frac{\partial UV}{\partial Y} \approx \frac{7.64 - 1.41}{5} = \frac{6.13}{5} = 1.246 \frac{UV}{Y}
\]
or
\[
\frac{\partial UV}{\partial Y} \approx \frac{10.82 - 7.64}{5} = \frac{3.28}{5} = .636 \frac{UV}{Y}
\]
or
\[
\frac{\partial UV}{\partial Y} \approx \frac{10.82 - 1.41}{10} = \frac{9.41}{10} = .941 \frac{UV}{Y}
\]
and I will use the last one, i.e. the \( \Delta Y \) of 10, though the others are surely OK.

B. Looking down to \(-70^\circ\), (smaller numbers since bigger numbers but with minus signs) I get:
\[
\frac{\partial UV}{\partial Y} \approx \frac{11.46 - 6.36}{10} = \frac{5.1}{10} = .51 \frac{UV}{Y}
\]

C. and so
\[
\frac{\partial}{\partial L} \left( \frac{\partial UV}{\partial Y} \right) = \frac{\partial^2 UV}{\partial L \partial Y} \approx \frac{\Delta \frac{\partial UV}{\partial Y}}{\Delta L} = \frac{.941 - .51}{10} = \frac{.4310}{10} = .0431 \frac{UV}{L}
\]

IV. There’s another one, have good cheer, this is the last:
\[
\frac{\partial^2 UV}{\partial Y \partial L} = \frac{\partial}{\partial Y} \left( \frac{\partial UV}{\partial L} \right)
\]
A. Left (High)
\[
\frac{\partial UV}{\partial L} \approx \frac{10.82 - 6.80}{-60 - (-80)} = \frac{-4.02}{20} = -.2010 \frac{UV}{L}
\]
B. Right (Low)

\[
\frac{\partial U V}{\partial L} \approx \frac{7.64 - 6.71}{-60 - (-80)} = \frac{.93}{20} = .0465 \frac{U V}{L}
\]

C. Thus

\[
\frac{\partial^2 U V}{\partial Y \partial L} \approx \frac{.2010 - .0465}{2005 - 2000} = \frac{.1549}{5} = .0309 \frac{U V}{Y - L}
\]

And our two estimates for the crosspartials are .0431 and .0309, which, given the accuracy of our table is good ’nuff. If I needed a single answer \( \frac{1}{2} (0.0431 + 0.0309) = 0.037 \) would do just fine. Any of the above calculations represents a estimate for the partial involved, and there are several ways to make each of these estimates. That wasn’t so bad, now was it??