Section 4.4 – Synthetic Division; The Remainder and Factor Theorems

Objectives
- Use the division algorithm to divide two polynomials by long division.
- Use the Remainder Theorem.
- Use the Factor Theorem.
- Factor polynomials and sketch their complete graph.

Preliminaries
The Division Algorithm states if \( f(x) \) and \( d(x) \) are polynomial functions with \( d(x) \neq 0 \) and the degree of \( d(x) \) less than the degree of \( f(x) \), then \( f(x) \) can be written in the following way:

\[ f(x) = \frac{\text{______________________________}}{d(x)} \]

Alternatively, we can write

\[ \frac{f(x)}{d(x)} = \frac{\text{______________________________}}{d(x)} \]

The Remainder Theorem states if a polynomial \( f(x) \) is divided by \( x - c \), then the remainder is \( \text{______________________________} \).

The Factor Theorem states the polynomial \( x - c \) is a factor of the polynomial \( f(x) \) if and only if \( \text{______________________________} \).

Warm-up

6. Divide the following using long division.

(A) 84 ÷ 13

(B) 804 ÷ 13
When dividing $f(x)$ by $d(x)$, $f(x)$ can be written as $f(x) = d(x)q(x) + r(x)$. What names can be given to the functions $d(x)$, $q(x)$, and $r(x)$?

4.4.1 Given the polynomials $f(x)$ and $d(x)$, find polynomials $q(x)$ and $r(x)$ by using long division. In these problems, you will compute $f(x) ÷ d(x)$ and write $f(x)$ as $f(x) = d(x)q(x) + r(x)$.

(A) $f(x) = 4x^3 - 6x^2 + 10x + 9$ and $d(x) = 2x - 1$
(B) \( f(x) = 2x^3 + 3x^2 - 7x + 11 \) and \( d(x) = x^2 + 3 \)

4.4.2 Use synthetic division to divide \( f(x) \) by \( x - c \), and then write \( f(x) \) as \( f(x) = (x - c)q(x) + r(x) \) for \( f(x) = x^3 + 5x^2 - 7 \) divided by \( x + 4 \).
4.4.3 Use the Remainder Theorem to determine the remainder when \( f(x) \) is divided by \( x - c \).

(A) \( f(x) = -3x^4 - 8x^3 + 5x + 11 \) divided by \( x + 2 \)

(B) \( f(x) = 2x^3 - 5x^2 - 3x + 9 \) divided by \( x - 1 \)
4.4.4 Determine whether the following polynomials of the form $x - c$ are factors of $f(x)$ by using the Factor Theorem. Check your results by using another method.

(A) Is $x + 4$ a factor of $f(x) = -3x^3 - 10x^2 + 6x - 8$?

(B) Is $x - 3$ a factor of $f(x) = x^4 - 11x^2 + 5x + 3$?
4.4.5 Determine the value of the constant $k$ so that $x - 2$ is a factor of the function $g(x) = -x^3 + kx^2 - 4x + 10$.

? How can we use the graph of a polynomial to factor the polynomial completely?

4.4.6 Factor $R(x) = 2x^3 + 5x^2 - x - 6$ completely given that $x - 1$ is a factor of $R(x)$. Sketch a graph of $R(x)$. 
4.4.7 Use your graphing calculator to graph \( T(x) = 4x^3 + 16x^2 - 23x - 15 \). Use long division to determine a complete factorization of \( T(x) \). Verify your results by comparing to the graph.

4.4.8 Use your graphing calculator to graph \( P(x) = -3x^4 + 12x^3 - 27x^2 + 60x - 60 \). Use long division to determine a complete factorization of \( P(x) \).
Section 4.4 Self-Assessment (Answers on page 256)

1.  \textbf{(Multiple Choice)} Determine the remainder when \(x^3 + 4x^2 - x + 7\) is divided by \(x^2 - 2x + 3\).

   \(\begin{align*}
   (A) \quad & x + 6 \\
   (B) \quad & -2x + 13 \\
   (C) \quad & 8x - 11 \\
   (D) \quad & 14x + 25 \\
   (E) \quad & \text{None of these}
   \end{align*}\)

2. Use the Remainder Theorem to determine the remainder when \(2x^4 - 7x^3 + 5x - 13\) is divided by \(x - 4\).

3.  \textbf{(Multiple Choice)} After dividing \(f(x) = x^3 + 5x + 2\) by \(x + 2\) and writing the answer in the form \(f(x) = (x + 2)q(x) + r(x)\), what is the formula for \(q(x)\)?

   \(\begin{align*}
   (A) \quad & q(x) = x + 3 \\
   (B) \quad & q(x) = x^2 + 2x + 9 \\
   (C) \quad & q(x) = x^2 - 2x + 9 \\
   (D) \quad & q(x) = x + 7 \\
   (E) \quad & q(x) = x^2 + 3x - 4
   \end{align*}\)

4. Use your graphing calculator to graph \(T(x) = 2x^3 - 19x^2 + 33x + 54\). Use your graph and division to determine a complete factorization of \(T(x)\).

5.  \textbf{(Multiple Choice)} Determine the value of \(k\) that makes \(x - 1\) a factor of the function \(f(x) = -5x^3 + 6x^2 + kx - 4\). The value of \(k\) is:

   \(\begin{align*}
   (A) \quad & \text{More than 7.5} \\
   (B) \quad & \text{Between 5.5 and 7.5} \\
   (C) \quad & \text{Between 3.5 and 5.5} \\
   (D) \quad & \text{Between 1.5 and 3.5} \\
   (E) \quad & \text{Less than 1.5}
   \end{align*}\)