Problem Sheet 1 (Jul. 30)

1) Solve the equation $5x \mod 123 = 7$.

2) In $GF(49)$, let $\alpha$ be a root of $-1$. Calculate $1/\alpha + \alpha$.
   GAP Commands: X, Roots0fUPol

3) Explain the output of the following commands. How could you evaluate $f$ at 1?

   gap> x:=X(Rationals,"x");
   x
   gap> f:=x^2+x+1;
   x^2+x+1
   gap> x:=1;
   1
   gap> f;
   x^2+x+1

4) Create a list of the first 20 cubes.
   GAP Commands: List, [1..20]

5) Create – with minimal GAP-assignments – the list $[1,2,8,7,17,5,4,3,9,10]$.

6) Determine all Elements $x \in GF(32)$ such that $x^5 + x^2 + 1 = 0$.
   GAP Commands: Filtered, GF, Z(2)^0

7) Determine all generators of $GF(27)^*$.

8) A number $n$ is called perfect if the sum of its divisors (denoted by $\sigma(n)$) equals $\sigma(n) = 2n$. Find all perfect numbers up to $10^6$.
   GAP Commands: Filtered, Sigma

9) The command Combinations([1..5]) returns all subsets of the set $\{1,\ldots,5\}$.
   a) Find all subsets of $\{1,\ldots,15\}$, whose entries add up to 15.
      GAP Commands: Filtered, Sum, Combinations
   b) Repeat the problem with subsets of $\{1,\ldots,22\}$ which sum up to 22. What happens? Can you modify the command to avoid the problem?
      GAP Commands: Combinations([1..22],i) for $1 \leq i \leq 22$, List, Concatenation.
10) Let \( A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \). Determine all solutions to the system of equations \( x \cdot A = (10, 10, 10) \) and \( A \cdot y = (-10, -11, -12)^T \).

**GAP Commands:** SolutionMat, NullspaceMat, TransposedMat

10a) A string in GAP is simply a list of characters and can be manipulated accordingly. Thus for example

```gap
s := "TUCSON";
Arrangements(s, 3);
```

returns all 3-letter sequences that can be made from the letters in the word “TUCSON”.

a) Read in the file `words.g`. This file defines a variable `words`, which is a list of strings for English words. Using this list, find all 3-letter combinations of “TUCSON”, which are proper English words.

b) We now want to find all words that can be made from the letters of “CANTALOUPE”, but already the 4-letter combinations take rather long to test. (You can use the command `time;` after a command to find out how many milliseconds it took.) Now issue the commands

```gap
words := List(words, Immutable);
IsSSortedList(words); # tests that the list is sorted.
```

and repeat the commands. Why is it suddenly so much faster?

c) (Once you know a bit more about groups in GAP) The number of arrangements of a given length (you can calculate this by `NrArrangements(s, k);`, e.g.) goes up substantially. Instead use `Combinations` and then run over all permutations in \( S_n \) using `Enumerator(SymmetricGroup(k))` (which does not write down all elements and takes little memory). Using this, determine how many English words can be formed from the letters of “TOBEORNOTTOBETHATISTHEQUESTION”.

11) Let

\[ m := \text{GeneratorsOfGroup(Centre(GL(2,5)))}[1]; \]

Create a new matrix, in which the last row is replaced by its negative.

**GAP Commands:** ShallowCopy

12) Write a function that adds the odd-position entries of a list. Apply it to \([1..100], [5, 3, 7] \) and [].

13) We define the Tribonacci numbers by \( T_0 = 1 \), \( T_1 = 3 \) and \( T_n = T_{n-1} + T_{n-2} \) for \( n > 1 \).

a) Write a function to calculate these numbers and use it to calculate \( T_{10000} \).

b) Determine the last 2 digits of \( T_{10^{10000}} \).

**GAP Commands:** PowerMod
The following function was written to create all vectors of length 3 with entries in \{0,1\}. But what does it create? Why? Can you fix the code?

```
1:=[];
# create vectors recursively, by trying out all entries at position ‘pos’:
allVectors:=function(v,pos)
local i;
    for i in [0..1] do
        v[pos]:=i;
        if pos=3 then
            Add(l,v);
        else
            allVectors(v,pos+1);
        fi;
    od;
end;

allVectors([0,0,0],1);
l;
```