There are five (5) problems on this exam, one on each page. They are each worth 20 points, but they are not all the same difficulty. We recommend you look over the entire exam before beginning to choosing which problem to start with. You are not necessarily expected to finish everything, but you should do as much as you can.

It is extremely important to show your work and explain your reasoning. If you need additional paper, raise your hand and the proctor will bring you some.

I will write on the board a number of integrals that you may find useful. Remind me if I forget!

NO CALCULATORS ARE PERMITTED ON THIS EXAM! If you obtain a numerical expression you cannot evaluate without a calculator, represent it by a suitable symbol throughout the rest of the problem.

The acronym "DE" will stand for "differential equation" throughout this exam.

Problem (1):

Problem (2):

Problem (3):

Problem (4):

Problem (5):

Whole Exam:
(1)(20 points) Consider the differential equation \( y' + 2y = t + e^t + \sin t. \)

(a)(17 points) Verify that
\[
y(t) = Ce^{mt} + A + Bt + De^t + E \cos t + F \sin t\]
is an exact solution for arbitrary constant \( C \) and certain specific values of \( m, A, B, D, E, F \) which you should find.

(b)(3 points) What value of \( C \) makes \( y(t = 0) = 0 \)?
A cylindrical tank of water, with cross-sectional area \( A \), empties a hole in the bottom at rate depending on the depth. Torricelli’s Constitutive Relation states that the exit velocity \( v \) equals \( \sqrt{2gh} \), where \( h(t) \) is the depth and \( g \) is the acceleration due to gravity. If the area of the hole is \( a \), the rate at which water flows out is \( cav \), where \( c \) is a ”stream contraction” factor. The resulting differential equation for the depth \( h(t) \) is

\[
A \frac{dh}{dt} = -ca\sqrt{2gh}.
\]

(a) (2 points) My head is spinning from all the physical parameters! Replace \( A, a, c, g, \) and 2 by 1; what is the resulting differential equation?

(b) (3 points) Find an equilibrium solution.

(c) (5 points) Find the general solution involving one arbitrary constant.

(d) (2 points) For what value of the constant is the initial depth \( h(t = 0) = 1 \)?

(e) (3 points) What is the depth at subsequent times \( t = 1, 2, 3 \)?

(f) (5 points) Describe the roles of the equilibrium solution and the general solution in determining the depth \( h(t) \) at any time \( t \) in the future.
(3)(20 points) The quantum theory of phonons in crystal lattices leads to the Schrödinger equation

\[- \frac{d^2 \psi}{dx^2} + x^2 \psi = E \psi,\]

where \( E \) is the energy of the state.

(a)(10 points) Verify that the ground state \( \psi(x) = e^{-x^2/2} \) is a solution, for a certain value of \( E \).

(b)(10 points) Verify that the 2-phonon state \( \psi(x) = (2x^2 - 1)e^{-x^2/2} \) is also a solution, for a different value of \( E \).

(You should find the values of \( E \) of course, in both (a) and (b).)
(4)(20 points) Consider the differential equation

\[(x^2 + 2)\frac{dz}{dx} = x^2 - 4xz.\]

(a)(10 points) Find the general solution of this differential equation, involving one arbitrary constant.

(b)(3 points) Find the value of the arbitrary constant in your solution to (a), for which \(z(x = 0) = 1\).

(c)(7 points) Find the general solution, involving one arbitrary constant, of the differential equation

\[\frac{dT}{dt} + 2T = \cos(3t).\]
(5) (20 points) Consider the differential equation \( \frac{du}{dt} = u^2 + 5u + 6 \).

(a) (5 points) Find all equilibrium solutions of this DE.

(b) (8 points) Find the general explicit solution of the DE, involving one arbitrary constant.

(c) (5 points) What value of the arbitrary constant makes \( u(t = 0) = 0 \)?

(d) (2 points) At what time \( t \) does the solution in (c) cease to exist?