1. Let $R$ be a commutative ring with $1 \neq 0$.

   (a) Prove that the nilradical of $R$ is equal to the intersection of the prime ideals of $R$. Hint: it’s easy to show using the definition of prime that the nilradical is contained in every prime ideal. Conversely, suppose that $f$ is not nilpotent and consider the set $S$ of ideals $I$ of $R$ with the property that “$n > 0 \implies f^n \not\in I$.” Show that $S$ has maximal elements and that any such maximal element must be a prime ideal.

   (b) Suppose that $R$ is reduced, i.e. that the nilradical of $R$ is the zero ideal. If $p$ is a minimal prime ideal of $R$, show that the localization $R_p$ has a unique prime ideal and conclude that $R_p$ is a field.

   (c) Again supposing $R$ to be reduced, prove that $R$ is isomorphic to a subring of a direct product of fields.

2. Let $R$ be a commutative ring with $1 \neq 0$ and let $\varphi : R \to R$ be a ring homomorphism. If $R$ is noetherian and $\varphi$ is surjective, show that $\varphi$ must be injective too, and hence an isomorphism. (Hint: Consider the iterates of $\varphi$ and their kernels.) Can you give a counter-example to this when $R$ is not noetherian?

3. As usual, for a prime $p$ we write $F_p = \mathbb{Z}/p\mathbb{Z}$ for the field with $p$ elements.

   (a) Find all monic irreducible polynomials in $F_p[X]$ of degree $\leq 3$ for $p = 2, 3, 5$.

   (b) Prove that for $f \in F_p[X]$ monic and irreducible, the ideal $(f(X))$ is maximal and hence that $F_p[X]/(f(X))$ is a field. Show that $F_p[X]/(f(X))$ has finite cardinality $p^{{\deg} f}$ and use part (3a) to explicitly construct finite fields of orders 8, 9, 25, 125.

   (c) Prove that $F_7[X]/(X^2 + 2)$ and $F_7[X]/(X^2 + X + 3)$ are both finite fields of size 49. Show that these fields are isomorphic by exhibiting an explicit isomorphism between them.

4. Let $R$ be a ring with $1 \neq 0$ and $M$ an $R$-module. Show that if $N_1 \subseteq N_2 \subseteq \cdots$ is an ascending chain of submodules of $M$ then $\cup_{i \geq 1} N_i$ is a submodule of $N$. Show by way of counterexample that modules over a ring need not have maximal proper submodules (in contrast to the special case of ideals in a ring with 1).

5. Let $R$ be any commutative ring with $1 \neq 0$ and $M$ and $R$-module. Show that the canonical map

   $\text{Hom}_R(R, M) \to M$

   sending $\varphi$ to $\varphi(1)$ is an isomorphism of $R$-modules.

6. Let $F = \mathbb{R}$ and let $V = \mathbb{R}^3$. Consider the linear map $\varphi : V \to V$ given by rotation through an angle of $\pi/2$ about the $z$-axis. Consider $V$ as an $F[X]$-module by defining

   $$(a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0)v := (a_n \varphi^n + a_{n-1} \varphi^{n-1} + \cdots + a_1 \varphi + a_0)v,$$

   where $\varphi^i$ is the composition of $\varphi$ with itself $i$-times.
(a) What are the $F[X]$-submodules of $V$?
(b) Show that $V$ is naturally a module over the quotient ring $F[X]/(X^3 - X^2 + X - 1)$.

7. Let $R$ be a ring with $1 \neq 0$.
   (a) For a left ideal $I$ of $R$ and an $R$-module $M$, define
   $$ IM := \{ r_1 m_1 + r_2 m_2 + \cdots + r_k m_k : r_i \in R, m_i \in M, k \in \mathbb{Z}_{\geq 0} \}.$$
   Show that $IM$ is an $R$-submodule of $M$.
   (b) Prove that for any ideal $I$ of $R$ and any positive integer $n$, there is a canonical isomorphism of $R$-modules
   $$ R^n/IR^n \simeq R/IR \times R/IR \times \cdots \times R/IR $$
   with $n$-factors in the product on the right.
   (c) Suppose now that $R$ is commutative and that $R^n \simeq R^m$ as $R$-modules. Show that $m = n$. Hint: reduce to the case of finite dimensional vector spaces over a field by applying (7b) with $I$ a maximal ideal of $R$.
   (d) If $R$ is commutative and $A$ is any finite set of cardinality $n$, show that $F(A) \simeq R^n$ as $R$-modules (Hint: Show that $R^n$ satisfies the same universal mapping property as $F(A)$ and deduce from this that one has maps in both directions whose composition in either order must be the identity). Conclude that the rank of a free module over a commutative ring is well-defined if it is finite.

8. Let $R$ be a ring with $1 \neq 0$ and $M$ an $R$-module. We say that $M$ is irreducible if $M \neq 0$ and the only submodules of $M$ are 0 and $M$.
   (a) Show that $M$ is irreducible if and only if $M$ is a nonzero cyclic $R$-module.
   (b) If $R$ is commutative, show that $M$ is irreducible if and only if $M \simeq R/I$ as $R$-modules for some maximal ideal $I$ of $R$.
   (c) Prove Schur’s lemma: if $M_1$ and $M_2$ are irreducible $R$-modules then any nonzero $R$-module homomorphism $\phi : M_1 \to M_2$ is an isomorphism.