1. Let $K/F$ be a degree 2 extension of fields.
   (a) If the characteristic of $F$ is not 2, prove that $K = F(a)$ for some $a \in K \setminus F$ with $a^2 \in F$.
   (b) Give a counterexample to (1a) if $F$ has characteristic 2.
   (c) Fix $F$ of characteristic not 2 and let $K_1, K_2$ be quadratic extensions of $F$ with $K_1 = F(a_1)$ and $K_2 = F(a_2)$ where $a_1^2 = b_1 \in F$. Prove that $K_1 \cong K_2$ as extensions of $F$ (i.e. that there exists an isomorphism of fields $K_1 \cong K_2$ restricting to the identity on $F$) if and only if $b_1/b_2 \in (F^\times)^2$ is a square. Conclude that the isomorphism classes of quadratic extensions of $F$ are in bijection with the group $F^\times/(F^\times)^2$.
   (d) Using (1c), give a complete list (without repetition) of all isomorphism classes of quadratic extensions of $\mathbb{Q}$.

2. For $a \in \mathbb{F}_p$, set $f_a(x) := X^p - X - a \in \mathbb{F}_p[X]$.
   (a) If $a = 0$, show that $f_a(X) = \prod_{u \in \mathbb{F}_p} (X - u)$.
   (b) Suppose that $a \neq 0$ and let $E_a$ be a splitting field of $f_a(X)$. If $r_1, r_2 \in E_a$ are roots of $f_a$, prove that $r_1 - r_2 \in \mathbb{F}_p$.
   (c) Show that $f_a(X)$ is irreducible for all $a \in \mathbb{F}_p^\times$.
   (d) Prove that $f_b(X)$ splits completely over $E_a$ for each fixed $a \in \mathbb{F}_p^\times$ and all $b \in \mathbb{F}_p^\times$. Conclude that $E_a$ is independent of $a$.

3. Find the minimal polynomials of $2 \cos(2\pi/5)$ and $2 \cos(2\pi/7)$ over $\mathbb{Q}$.

4. For each of the following extensions, determine $[K : F]$ and an $F$-basis of $K$.
   (a) $F = \mathbb{Q}$, $L = \mathbb{Q}(a, b)$ with $a^2 = 6$ and $b^3 = 2$.
   (b) $F = \mathbb{C}(T)$ and $L$ is the splitting field of $X^n - T$ over $F$, with $n$ a positive integer.
   (c) $F = \mathbb{F}_p(T)$ and $L$ is the splitting field of $X^p - T$ over $F$, with $p$ a prime.

5. Let $K/F$ be a finite extension of fields and let $\alpha \in K$. Then $\alpha$ induces an $F$-linear map of finite-dimensional $F$-vector spaces $m_\alpha : K \to K$.
   (a) Prove that $\alpha$ is a root of the characteristic polynomial of the linear map $m_\alpha$. Hint: select a suitable $F$-basis of $F(\alpha)$.
   (b) Use (5a) to find a monic degree 3 polynomial with $\mathbb{Q}$-coefficients satisfied by $1 + \sqrt[3]{2} + \sqrt[3]{4}$.
   (c) Prove that if $K = F(\alpha)$, then the characteristic polynomial of $m_\alpha$ as a linear map $K \to K$ is in fact the minimal polynomial of $\alpha$ over $F$.

6. For each of the following algebraic elements $\alpha$ of the given field extension $K/\mathbb{Q}$, express $1/\alpha$ and $1/(\alpha + 1)$ as polynomials in $\alpha$ with $\mathbb{Q}$-coefficients.
(a) $K$ is the splitting field of $f = X^3 - 3X + 1$ and $\alpha$ is a root of $f$.
(b) $K$ is the splitting field of $f = X^4 + X^3 + x^2 + x + 1$ and $\alpha$ is a root of $f$.
(c) $K$ is the splitting field of $f = X^5 - 3X + 3$ and $\alpha$ is a root of $f$.

7. Prove that $X^4 - 5$ is irreducible over $\mathbb{Q}$ and has splitting field $K$ of degree 8 over $\mathbb{Q}$. Describe this splitting field explicitly as $\mathbb{Q}(a, b)$ where $a$ is a root of $X^4 - 5$ and $b^2 \in \mathbb{Q}$. In terms of $a$ and $b$, write down a $\mathbb{Q}$-basis for $K$.

8. Describe the splitting fields of $f := X^3 - 5$ over $\mathbb{F}_{11}$ and $\mathbb{F}_7$ and factor $f$ into linear factors over each extension.