ON THE $U_p$ OPERATOR IN CHARACTERISTIC $p$

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Abstract. For a perfect field $\kappa$ of characteristic $p > 0$, a positive integer $N$ not divisible by $p$, and an arbitrary subgroup $\Gamma$ of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$, we prove (with mild additional hypotheses when $p \leq 3$) that the $U$-operator on the space $M_k(\mathcal{P}_T/\kappa)$ of (Katz) modular forms for $\Gamma$ over $\kappa$ induces a surjection $U : M_k(\mathcal{P}_T/\kappa) \to M_{k'}(\mathcal{P}_T/\kappa)$ for all $k \geq p + 2$, where $k' = (k - k_0)/p + k_0$ with $2 \leq k_0 \leq p + 1$ the unique integer congruent to $k$ modulo $p$. When $\kappa = \mathbb{F}_p$, $p \geq 5$, $N \neq 2, 3$, and $\Gamma$ is the subgroup of upper-triangular or upper-triangular unipotent matrices, this recovers a recent result of Dewar [3].

1. Introduction

Fix a prime $p$, an integer $N > 0$ with $p \nmid N$, and a subgroup $\Gamma$ of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Let $\tilde{\Gamma}$ be the preimage in $\text{SL}_2(\mathbb{Z})$ of $\Gamma_0 := \Gamma \cap \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$, and write $\tilde{M}_k(\tilde{\Gamma})$ for the space of weight $k$ mod $p$ modular forms for $\tilde{\Gamma}$ (in the sense of Serre [8, §1.2]). When $N = 1$, a classical result of Serre [8, §2.2, Théorème 6] asserts that the $U_p$ operator is a contraction: for $k \geq p + 2$, the map $U_p : \tilde{M}_k(\Gamma(1)) \to \tilde{M}_{k'}(\Gamma(1))$ has image contained in $\tilde{M}_{k'}(\Gamma(1))$ for some $k' < k$ satisfying $pk' \leq k + p^2 - 1$. In fact, Serre’s result may be generalized and significantly sharpened:

Theorem 1.1. Let $\kappa$ be a perfect field of characteristic $p$ and denote by $M_k(\mathcal{P}_T/\kappa)$ the space of weight $k$ Katz modular forms for $\Gamma$ over $\kappa$ (see §3). Let $k_0$ be the unique integer between 2 and $p + 1$ congruent to $k$ modulo $p$, and if $p \leq 3$, assume that $N > 4$ and that $\Gamma_0$ is a subgroup of the upper-triangular unipotent matrices. Then for $k \geq p + 2$, the $U$-operator (see §3) acting on $M_k(\mathcal{P}_T/\kappa)$ induces a surjection $U : M_k(\mathcal{P}_T/\kappa) \to M_{k'}(\mathcal{P}_T/\kappa)$, for $k' := (k - k_0)/p + k_0$.

When $\tilde{\Gamma} = \Gamma_*(N)$ for $* = 0, 1$ and $\kappa = \mathbb{F}_p$, the endomorphism $U$ coincides with the usual Atkin operator $U_p$ (see Corollary 3.3). In particular, if $p \geq 5$, so (by Theorems 1.7.1, and 1.8.1–1.8.2 of [5]) $\tilde{M}_k(\tilde{\Gamma}) \simeq M_k(\mathcal{P}_T/\mathbb{F}_p)$ and $N \neq 2, 3$, Theorem 1.1 is due to Dewar [3]. The proofs of both Serre’s original result and Dewar’s refinement of it rely on a delicate analysis of the interplay between the operators $U_p$, $V_p$, and $\theta$ acting on mod $p$ modular forms. In the present note, we take an algebro-geometric perspective, and show how Theorem 1.1 follows immediately from a (trivial extension of a) general theorem of Tango [9] on the behavior of vector bundles under the Frobenius map. In this optic, the contractivity of $U_p$ in characteristic $p$ is simply an instance of the “Dwork Principle” of analytic continuation along Frobenius. In particular, we use neither the $\theta$-operator, nor the notion of “filtration” of a mod $p$ modular form. Moreover, our formulation of Theorem 1.1 and its proof totally avoid the use of $q$-expansions, so should be readily adaptable to the Shimura curve setting.

2. Tango’s Theorem

Fix a perfect field $\kappa$ of characteristic $p$, and write $\sigma : \kappa \to \kappa$ for the $p$-power Frobenius automorphism of $\kappa$. Let $X$ be a smooth, proper, and geometrically connected curve over $\kappa$ of genus $g$. Attached to
$X$ is its Tango number:

$$n(X) := \max \left\{ \sum_{x \in X(\overline{\kappa})} \frac{\text{ord}_x(df)}{p} : f \in \overline{\kappa}(X) \setminus \overline{\kappa}(X)^p \right\},$$

where $\overline{\kappa}(X)$ is the function field of $X_{\overline{\kappa}}$. As in Lemma 10 and Proposition 14 of [9], it is easy to see that $n(X)$ is a well-defined integer satisfying $-1 \leq n(X) \leq \lfloor (2g - 2)/p \rfloor$, with the lower bound an equality if and only if $g = 0$.

**Proposition 2.1 (Tango).** Let $S \neq X$ be a reduced closed subscheme of $X$ with ideal sheaf $\mathcal{I}_S \subseteq \mathcal{O}_X$, and let $\mathcal{L}$ be a line bundle on $X$. If $\deg \mathcal{L} > n(X)$ then the natural $\sigma$-linear map

$$(2.2) \quad F^* : H^1(X, \mathcal{L}^{-1} \otimes \mathcal{I}_S) \longrightarrow H^1(X, \mathcal{L}^{-p} \otimes \mathcal{I}_S)$$

induced by pullback by the absolute Frobenius of $X$ is injective, and the natural $\sigma^{-1}$-linear “trace map”

$$(2.3) \quad F_* : H^0(X, \Omega^1_{X/\kappa} \otimes \mathcal{L}^p) \longrightarrow H^0(X, \Omega^1_{X/\kappa}(S) \otimes \mathcal{L})$$

given by the Cartier operator ([1], [7, §10]) is surjective.

**Proof.** The formation of (2.2) and (2.3) is compatible, via $\sigma$- (respectively $\sigma^{-1}$-) linear extension, with any scalar extension $\kappa \rightarrow \kappa'$ to a perfect field $\kappa'$; we may therefore assume that $\kappa$ is algebraically closed. When $g = 0$ we have $X \simeq \mathbb{P}^1$ and the proposition is easily verified by direct calculation, so we may further assume that $g > 0$. As the two assertions are dual by Serre duality [7, §10, Proposition 9], it suffices to prove the injectivity of (2.2). The case $S = \emptyset$ is Tango’s Theorem [9, Theorem 15]. In general, as $\deg(\mathcal{L}) > 0$ and $\mathcal{O}_X/\mathcal{I}^j_S$ is a skyscraper sheaf for all $j > 0$, one finds a commutative diagram with exact rows

$$\begin{array}{cccccc}
0 & \longrightarrow & H^0(X, \mathcal{O}_X/\mathcal{I}_S) & \longrightarrow & H^1(X, \mathcal{L}^{-1} \otimes \mathcal{I}_S) & \longrightarrow & H^1(X, \mathcal{L}^{-1}) & \longrightarrow & 0 \\
& & F^* & \downarrow & F^* & \downarrow & F^* & & \\
0 & \longrightarrow & H^0(X, \mathcal{O}_X/\mathcal{I}^p_S) & \longrightarrow & H^1(X, \mathcal{L}^{-p} \otimes \mathcal{I}^p_S) & \longrightarrow & H^1(X, \mathcal{L}^{-p}) & \longrightarrow & 0 \\
& & & \downarrow & & & \downarrow & & \\
0 & \longrightarrow & H^0(X, \mathcal{O}_X/\mathcal{I}_S) & \longrightarrow & H^1(X, \mathcal{L}^{-p} \otimes \mathcal{I}_S) & \longrightarrow & H^1(X, \mathcal{L}^{-p}) & \longrightarrow & 0
\end{array}$$

in which the lower vertical arrows are induced by the inclusion $\mathcal{I}^p_S \subseteq \mathcal{I}_S$. Using that $\kappa = \overline{\kappa}$ and identifying $H^0(X, \mathcal{O}_X/\mathcal{I}_S)$ with $\kappa^S$, the left vertical composite is easily seen to coincide with the map $\oplus_S \sigma : \kappa^S \rightarrow \kappa^S$ which is $\sigma$ on each factor; it is therefore injective. As the right vertical composite map is injective by Tango’s Theorem, an easy diagram chase finishes the proof. \hfill \blacksquare

3. Modular forms mod $p$ as differentials on the Igusa curve

In order to apply Tango’s Theorem to prove Theorem 1.1, we must recall Katz’s geometric definition of mod $p$ modular forms, and Serre’s interpretation of them as certain meromorphic differentials on the Igusa curve.

Let us write $^3 R_\Gamma := (\mathbb{Z}[\zeta_N])^{\det(\Gamma)}$, and for any $R_\Gamma$-algebra $A$ denote by $\mathcal{O}_R/A$ the moduli problem $([\Gamma(N)]/\Gamma)^{R_\Gamma-\text{can}} \otimes_R A$ on $(\text{Ell}/A)$ (see §3.1, §7.1, 9.4.2, and 10.4.2 of [6]) and by $M_\kappa(\mathcal{O}_R/A)$ the

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Footnotes:

2Note that $\kappa$-linear duality interchanges $\sigma$-linear maps with $\sigma^{-1}$-linear ones.

3Here, we follow the notation of [6, §9.4]: By definition $\mathbb{Z}[\zeta_N]$ is the finite free $\mathbb{Z}$-algebra $\mathbb{Z}[X]/\Phi_N(X)$, where $\Phi_N$ is the $N$-th cyclotomic polynomial and $\zeta_N$ corresponds to $X$, equipped with its natural Galois action of $(\mathbb{Z}/N\mathbb{Z})^\times$. 
space of weight \( k \) Katz modular forms for \( \mathcal{S}_1 / \mathbb{A} \) (e.g. \([10, \S 6]\)) that are holomorphic at \( \infty \) in the sense of \([5, \S 1.2]\). Equivalently, \( M_k(\mathcal{S}_1 / \mathbb{A}) \) is the \( \mathbb{A} \)-submodule of level \( N \), weight \( k \) modular forms in the sense of \([2, \text{VII.3.6}]\) that are invariant under the natural action of \( \Gamma_0 := \Gamma \cap \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \). Viewing \( C \) as an \( R_\Gamma \)-algebra via \( \zeta \mapsto \exp(2\pi i / N) \), we note that \( M_k(\mathcal{S}_1 / C) \) is the “classical” space of weight \( k \) modular forms for \( \Gamma \) over \( C \) defined via the transcendental theory \([2, \text{VII.4}]\).

Now fix a ring homomorphism \( R_\Gamma \to \kappa \) with \( \kappa \) a perfect field of characteristic \( p \). From here until the end of this section we will assume that \( \mathcal{S}_1 / \kappa \) is representable and that \(-1\) acts without fixed points on the space of cusp-labels for \( \Gamma \) (see \([6, \S 10.6]\) and c.f. \([6, 10.13.7–8]\)). We will later explain how to relax these hypotheses to those of Theorem 1.1. We write \( Y_\Gamma \) (respectively \( X_\Gamma \)) for the associated (compactified) moduli scheme; by \([6, 10.13.12]\), one knows that \( X_\Gamma \) is a proper, smooth, and geometrically connected curve over \( \kappa \). Writing \( \rho : \mathcal{I} \to Y_\Gamma \) for the universal elliptic curve, our hypothesis that \(-1\) acts without fixed points on the cusp labels for \( \Gamma \) ensures that the line bundle \( \omega_\Gamma := \rho_*\Omega^1_{\mathcal{I}/Y_\Gamma} \) on \( Y_\Gamma \) admits a canonical extension, again denoted \( \omega_\Gamma \), to a line bundle on \( X_\Gamma \) \([6, 10.13.4, 10.13.7]\). By definition, \( M_k(\mathcal{S}_1 / \kappa) = H^0(X_\Gamma, \omega^k_\Gamma) \).

Let \( I_\Gamma \) be the Igusa curve of level \( p \) over \( X_\Gamma \); by definition, \( I_\Gamma \) is the compactified moduli scheme associated to the simultaneous problem \([\mathcal{S}_1 / \kappa, [\text{Ig}(p)]](\text{Ell} / \kappa) \) \([6, \S 12]\). By \([6, 12.7.2]\), the Igusa curve is proper, smooth, and geometrically connected, and the natural map \( \pi : I_\Gamma \to X_\Gamma \), is finite étale and Galois with group \((\mathbb{Z} / p\mathbb{Z})^\times\) outside the supersingular points, and totally ramified over every supersingular point. Define \( \omega := \pi^*\omega_\Gamma \), and recall \([6, 12.8.2–3]\) there is a canonical section \( a \in H^0(I_\Gamma, \omega) \) which vanishes to order \( 1 \) at each supersingular point and on which \( d \in (\mathbb{Z} / p\mathbb{Z})^\times \) acts through \( \chi^{-1} \), for \( \chi : (\mathbb{Z} / p\mathbb{Z})^\times \to \mathbb{F}_p^* \to \mathbb{F}_p \) the mod \( p \) Teichmüller character. The following is a straightforward generalization of a theorem of Serre; see \([6, \S 12.8]\) and c.f. Propositions 5.7–5.10 of \([4]\).

Proposition 3.1. Fix an integer \( k \geq 2 \). For any integer \( k_0 \leq k \) with \( 2 \leq k_0 \leq p + 1 \), the map \( f \mapsto \pi^* f / a^{k_0 - 2} \) induces an natural isomorphism of \( \kappa \)-vector spaces

\[
M_k(\mathcal{S}_1 / \kappa) \simeq H^0(I_\Gamma, \Omega^1_{I_\Gamma / \kappa}(\text{cusps} + \delta_{k_0} \cdot \text{ss}) \otimes \omega^{k - k_0})(\chi^{k_0 - 2}),
\]

where \( \delta_{k_0} = 1 \) when \( k_0 = p + 1 \) and is zero otherwise; here, \( \text{ss} \) and \( \text{cusps} \) are the reduced supersingular and cuspidal divisors, respectively.

Proof. The proof is a straightforward adaptation of Propositions 5.7–5.10 of \([4]\); for the convenience of the reader, we sketch the argument. Thanks to \([6, 10.13.11]\), the Kodaira-Spencer map \([6, 10.13.10]\) provides an isomorphism \( \omega^2_\Gamma \simeq \Omega^1_{I_\Gamma / \kappa}(\text{cusps}) \) of line bundles on \( X_\Gamma \) which, after pullback along \( \pi \), gives an isomorphism

\[
\omega^2 \simeq \Omega^1_{I_\Gamma / \kappa}(-(p - 2)\text{ss} + \text{cusps})
\]

of line bundles on \( I_\Gamma \) as \( \pi \) is étale outside \( \text{ss} \) and totally (tamely) ramified at each supersingular point.

Since \( a \in H^0(I_\Gamma, \omega) \) has simple zeroes along \( \text{ss} \), via \((3.2)\) any global section \( f \) of \( \omega^k_\Gamma \) induces a global section \( \pi^* f / a^{k_0 - 2} \) of \( \Omega^1_{I_\Gamma / \kappa}(\text{cusps} + \delta_{k_0} \cdot \text{ss}) \otimes \omega^{k - k_0} \) on which \((\mathbb{Z} / p\mathbb{Z})^\times \) acts through \( \chi^{k_0 - 2} \); thus the map \((3.1)\) is well-defined. As \( \pi : I_\Gamma \to X_\Gamma \) is a degree \( p - 1 \) generically étale branched cover, the canonical trace map \( \pi_* \mathcal{O}_{I_\Gamma} \to \mathcal{O}_{X_\Gamma} \) of locally free \( \mathcal{O}_{X_\Gamma} \)-modules induces a trace mapping \( \pi_* : H^0(I_\Gamma, \omega^k) \to H^0(X_\Gamma, \omega^k_{\Gamma}) \) which satisfies \( \pi_* \pi^* = \text{deg} \, \pi = p - 1 \); it follows easily that \((3.1)\) is injective. To prove surjectivity, observe that by \((3.2)\), a global section of \( \Omega^1_{I_\Gamma / \kappa}(\text{cusps} + \delta_{k_0} \cdot \text{ss}) \otimes \omega^{k - k_0} \) gives a meromorphic section \( h \) of \( \omega^{k - k_0 + 2} \) satisfying \( \text{ord}_x(h) \geq -(p - 1) \) at each supersingular point \( x \), with equality possible only when \( k_0 = p + 1 \). If \( h \) lies in the \((k_0 - 2)\)-eigenspace of the action of \((\mathbb{Z} / p\mathbb{Z})^\times \), then \( f := a^{k_0 - 2} h \) descends to a meromorphic section of \( \omega^k_{\Gamma} \) over \( X_\Gamma \) satisfying

\[
(p - 1) \text{ord}_x(f) = \text{ord}_x(h) + k_0 - 2 \geq k_0 - p - 1
\]
at each supersingular point \( x \in X_\Gamma(\kappa) \), with equality possible only when \( k_0 = p + 1 \). Since the left side is a multiple of \( p - 1 \) and \( k_0 \geq 2 \), we must have \( \text{ord}_x(f) \geq 0 \) in all cases, and \( f \) is a global (holomorphic) section of \( \omega_\kappa^k \) over \( X_\Gamma \) with \( \pi^* f/\alpha^{k_0-2} = h \). ■

Using Proposition 3.1, the Cartier operator \( F_\kappa \) on meromorphic differentials induces, by “transport of structure”, a \( \sigma^{-1} \)-linear map \( U : M_k(\mathcal{P}/\kappa) \to M_k(\mathcal{P}/\kappa) \). If \( G \) is any group of automorphisms of \( X_\Gamma \), then the action of \( G \) commutes with \( F_\kappa \) (ultimately because the \( p \)-power map in characteristic \( p \) commutes with all ring homomorphisms), and we likewise obtain a \( \sigma^{-1} \)-linear endomorphism \( U \) of \( M_k(\mathcal{P}/\kappa) \). This allows us to define \( U \) even when \( \mathcal{P}/\kappa \) is not representable as follows. Choose a prime \( \ell > 3N \), and let \( \Gamma' \) be the unique subgroup of \( \text{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) \) projecting to the trivial subgroup of \( \text{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) \) and to \( \Gamma \) in \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \). Then for any perfect field \( \kappa' \) of characteristic \( p \) admitting a map from \( R_{\Gamma'} \), the moduli problem \( \mathcal{P}_{\Gamma}/\kappa' \) is representable, there is a natural action of \( G := \text{SL}_2(\mathbb{Z}/\ell\mathbb{Z}) \) on \( M_k(\mathcal{P}_{\Gamma}/\kappa') \), and one has \( M_k(\mathcal{P}_{\Gamma}/\kappa') = M_k(\mathcal{P}_{\Gamma}/\kappa')^G \) (c.f. [2, VII.3.3] and [5, §3.0.1]).

Via the canonical base-change isomorphism \( M_k(\mathcal{P}/\kappa) \otimes_\kappa \kappa' \simeq M_k(\mathcal{P}/\kappa') \), we obtain the desired endomorphism \( U \) of \( M_k(\mathcal{P}/\kappa) \) by descent, and it is straightforward to check that it is independent of our initial choices of \( \ell \) and \( \kappa' \). By post-composition with the \( \sigma \)-linear isomorphism \( M_k(\mathcal{P}/\kappa) \simeq M_k(\mathcal{P}^{-1}/\kappa) \) induced by the “exotic isomorphism” of moduli problems \( \mathcal{P}/\kappa \simeq \mathcal{P}_{\Gamma}/\kappa \) [6, 12.10.1] we obtain a \( \kappa \)-linear map \( U^\#: M_k(\mathcal{P}/\kappa) \to M_k(\mathcal{P}_{\Gamma}/\kappa) \). When \( \mathcal{P} \) is defined over \( \mathbb{F}_p \) in the sense that \( R_\Gamma \) admits a (necessarily unique) surjection to \( \mathbb{F}_p \), one has canonically \( \mathcal{P}/\mathbb{F}_p = \mathcal{P}_{\Gamma}/\mathbb{F}_p \) as problems on \( (\text{Ell}/\mathbb{F}_p) \), and \( U^\# \) is an endomorphism of \( M_k(\mathcal{P}/\mathbb{F}_p) \). The maps \( U \) and \( U^\# \) are natural generalizations of Atkin’s \( U_p \)-operator:

**Proposition 3.2.** Suppose that \( \mathcal{P}/\kappa \) is representable and let \( c \) be any cusp of \( X(\Gamma) \) defined over \( \kappa \). Then \( q^{1/e} \) is a uniformizing parameter at \( c \) for some divisor \( e \) of \( N \), and for \( f \in M_k(\mathcal{P}/\kappa) \), the formal expansions of \( Uf \) at \( c \) and of \( U^\# f \) at \( c^{-1} \) are:

\[
Uf = \sum_{n \geq 0} \sigma^{-1}(a_{np})q^{n/e} \quad \text{and} \quad U^\# f = \sum_{n \geq 0} a_{np}q^{n/e} \quad \text{respectively, for} \quad f = \sum_{n \geq 0} a_nq^{n/e}.
\]

**Proof.** Using the well-known local description of the Cartier operator on meromorphic differentials (e.g. [7, §10, Proposition 8]), the result follows easily from the arguments of Propositions 2.8 and 5.7 of [4]; see also (the proof of) [4, Proposition 5.9]. ■

**Corollary 3.3.** Suppose that \( \bar{\Gamma} = \Gamma_\star(N) \) for \( \star = 0, 1 \). Then \( R_\Gamma = \mathbb{Z} \) and the resulting endomorphisms \( U \) and \( U^\# \) of \( M_k(\mathcal{P}/\mathbb{F}_p) \) coincide with the Atkin operator \( U_p \), whether or not \( \mathcal{P}/\mathbb{F}_p \) is representable.

**Proof.** That \( R_\Gamma = \mathbb{Z} \) is clear, as \( \det(\Gamma) = (\mathbb{Z}/N\mathbb{Z})^\times \). By the discussion above, we may reduce to the representable case, and the result then follows from Proposition 3.2 and the \( q \)-expansion principle. ■

## 4. Proof of Theorem 1.1

We now prove Theorem 1.1. Fix \( k \) and let \( k_0 \) and \( k' \) be as in the statement of Theorem 1.1. First suppose that \( \mathcal{P}_\Gamma \otimes_{\mathbb{F}_p} \kappa \) is representable and that \( -1 \) acts without fixed points on the cusp-labels of \( \Gamma \). Using (3.2) and the fact that \( a \) has simple zeroes along \( \kappa \) as we compute (c.f. [6, 12.9.4])

\[
\deg \omega = \frac{2g - 2}{p} + \frac{1}{p} \deg(\text{cusps}) \geq \left\lceil \frac{2g - 2}{p} \right\rceil \geq n(I_\Gamma)
\]

\( ^4 \)Explicitly, this isomorphism sends \( f \in M_k(\mathcal{P}/\kappa) \) to the modular form \( f^\sigma \) defined by \( f^\sigma(E, \alpha) := f(E^\sigma, \alpha^\sigma) \).

\( ^5 \)A sufficient condition for this to happen is that \( \det(\Gamma) \) contain the residue class of \( p \bmod N \).
where $g$ is the genus of $I_\Gamma$. Applying Proposition 2.1 with $X = I_\Gamma$, $S = \text{cusps} + \delta_{k_0} \cdot \text{ss}$, and $\mathcal{L} = \omega$, we conclude from (2.3) and the relation $k - k_0 = p(k' - k_0)$ that the Cartier operator

$$F_\omega : H^0(I_\Gamma, \Omega^1_{I_\Gamma/\kappa}(S) \otimes \omega^{k-k_0}) \longrightarrow H^0(I_\Gamma, \Omega^1_{I_\Gamma/\kappa}(S) \otimes \omega^{k'-k_0})$$

is surjective whenever $k - k_0 \geq p$. Passing to $\chi^{k_0 - 2}$-eigenspaces for $(\mathbb{Z}/p\mathbb{Z})^\times$ and appealing to Proposition 3.1 and Corollary 3.3 then completes the proof in this case.

Now when $p \leq 3$, the hypotheses $N > 4$ and $\tilde{\Gamma} \subseteq \Gamma_1(N)$ of Theorem 1.1 ensure that $\mathcal{P}_\Gamma \otimes_R \kappa$ is representable (as it maps to the moduli problem $[\Gamma_1(N)]$, which is representable for $N \geq 4$ by [6, 10.9.6]) and that $-1$ acts without fixed points on the cusp-labels of $\Gamma$ [6, 10.7.4]. If $p \geq 5$, we may choose a prime $\ell > 3N$ with $\ell \not\equiv 0, \pm 1 \mod p$, so that $p \nmid |\text{SL}_2(\mathbb{Z}/\ell \mathbb{Z})|$. Then for $N' := N\ell$ and $\Gamma'$ the subgroup $1 \times \Gamma$ of $\text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) = \text{SL}_2(\mathbb{Z}/N\ell \mathbb{Z})$, we have (after passing to an appropriate extension $\kappa'$ of $\kappa$) that $\mathcal{P}_{\Gamma'} \otimes_{R_{\Gamma'}} \kappa'$ is representable with $-1$ acting freely on the cusp-labels of $\Gamma'$ [6, 10.7.1, 10.7.3]. We conclude that the $\tilde{U}$-operator induces a surjection of $\kappa[\text{SL}_2(\mathbb{Z}/\ell \mathbb{Z})]$-modules $M_k(\mathcal{P}_{\Gamma'/\kappa'}) \rightarrow M_{k'}(\mathcal{P}_{\Gamma'/\kappa'})$. Our choice of $\ell$ ensures that the ring $\kappa[\text{SL}_2(\mathbb{Z}/\ell \mathbb{Z})]$ is semisimple, so passing to $\text{SL}_2(\mathbb{Z}/\ell \mathbb{Z})$-invariants is exact. As the space of $\text{SL}_2(\mathbb{Z}/\ell \mathbb{Z})$-invariant weight $k$ modular forms for $\Gamma'$ coincides with $M_k(\mathcal{P}_{\Gamma'/\kappa'})$ (c.f. the definition of $U$ in §3), passing to invariants and descending from $k'$ to $\kappa$ then completes the proof of Theorem 1.1 in the general case.

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