Math 223

Disclaimer:
It is not a good idea to rely exclusively on reading through old exam solutions as a way to prepare for the final exam. In particular, this semester's course director may not have written any of the exams available from this page, so the ones he/she gives will almost certainly have a somewhat different flavor.

Topic Warning:
Because the topics taught differ slightly from semester to semester, it is not a good idea to use the old exams to gauge the content of the exams this semester.
1. (8) Consider two vectors $\mathbf{v} = 4\mathbf{i} - \mathbf{j} + a\mathbf{k}$ and $\mathbf{w} = a\mathbf{i} + 5\mathbf{j} - \mathbf{k}$. For what values of $a$ are $\mathbf{v}$ and $\mathbf{w}$ perpendicular?

2. (8) Find an equation for the plane passing through the points $(0, 2, 1)$, $(1, 1, 5)$, and $(2, 0, 11)$.

3. (8) Find a vector of magnitude 10 normal to the plane $5x + 3y = 6z + 1$. 
4. (8) At which point (or points) on the ellipsoid \( x^2 + 4y^2 + z^2 = 9 \) is the tangent plane parallel to the plane \( z = 0 \)?

5. (8) Find a parametric equation for a line through the point \((1, -3, 5)\) and parallel to the vector \(5\hat{i} + 3\hat{j} - \hat{k}\)
6. (8) Find the directional derivative of \( f(x, y) = x^2y + y^2x \) at the point \((1, -1)\) in the direction of \( 3\mathbf{i} - 4\mathbf{j} \).

7. (7) Compute the flux integral \( \int_S \mathbf{F} \cdot d\mathbf{A} \) where \( \mathbf{F} = \mathbf{i} + \mathbf{j} - \mathbf{k} \) and \( S \) is the surface \( z = x^2 - y^2, \; 0 \leq x \leq 3, \; 0 \leq y \leq 3 \), oriented upwards.
8. (20) Consider the integral $\int_{0}^{1} \int_{(8y)^{1/3}}^{2} \frac{1}{1+x^4} \, dx \, dy$.

(a) Interchange the order of integration. Show your work by including a sketch of the region of integration.

(b) Evaluate the integral.
9. (20) Let $\mathbf{F} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$. Evaluate the following:

(a) $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $C$ is the line from $(0,0,0)$ to $(1,1,1)$.

(b) $\int_S \mathbf{F} \cdot d\mathbf{A}$ where $S$ is the triangle in the plane $y = 10$ with vertices $(0,10,0)$, $(4,10,0)$, and $(0,10,1)$, oriented in the direction of increasing $y$.

(c) $\int_S \mathbf{F} \cdot d\mathbf{A}$ where $S$ is the sphere of radius 2 centered at $(5,5,0)$, oriented outward.
10. (15) Let $H(x, y, z) = \sin(2x + y) + z$. Find the equation of the tangent plane to the level surface $H(x, y, z) = 5$ at the point $(\pi, \pi, 5)$. 
11. (20) Let \( \vec{F} = (y + z)x \hat{i} + y \hat{j} + xyz \hat{k}. \)

(a) Find \( \text{curl}(\vec{F}) \).

(b) Let \( S \) be the surface \( x^2 + y^2 + z = 25 \), with \( 0 \leq z \leq 25 \), oriented upward. Find the value of the flux integral \( \int_S \text{curl}(\vec{F}) \cdot d\vec{A} \).
12. (15) Let \( a \) be a constant, \( a \neq 2 \), and consider function \( f(x, y) = \frac{1}{2}x^2 + 2y + 2xy + ay^2 \).

(a) Find the critical point of \( f \).

(b) Find all values of \( a \) so that the critical point is a global minimum.
13. (20) Consider the contour diagram for the function \( f(x, y) \) sketched below.

(a) Sketch a graph of \( f(x, 0) \).

(b) Determine whether the following quantities are positive, negative, or equal to zero.

\[
\begin{align*}
f_{xx}(0, 0) \text{ is } & \quad \text{ or } \\
f_{xy}(0, 0) \text{ is } & \quad \text{ or } \\
\end{align*}
\]

(c) If all contour lines are parallel to the line \( 2x + y = 0 \), then determine the direction in which the gradient of \( f \) points, as a unit vector.
14. (15) Rewrite the integral
\[ \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{0} \int_{\sqrt{18-x^2-y^2}}^{\sqrt{x^2+y^2}} xy \, dz \, dx \, dy \]
in spherical and cylindrical coordinates.

(a) In spherical coordinates, use the order of integration \( d\rho \, d\theta \, d\phi \).

(b) In cylindrical coordinates, use the order of integration \( dz \, dr \, d\theta \).
15. (20) Consider the 2-dimensional force field \( \overrightarrow{F} = 2xe^{x^2-5y} \mathbf{i} - 5e^{x^2-5y} \mathbf{j} \).

(a) Is \( \overrightarrow{F} \) conservative? If so, find a potential function \( f(x, y) \) whose gradient is \( \overrightarrow{F} \).

(b) Find the work done by the force field \( \overrightarrow{F} \) in moving an object from \( P(0, 2) \) to \( Q(-2, 0) \) along the path formed by \( C_1 \) followed by \( C_2 \) as shown in the figure below. \( C_1 \) and \( C_2 \) may be parametrized as follows:

\[
\begin{align*}
C_1 & : \quad x = t, \quad y = 2 - t, \quad 0 \leq t \leq 2, \\
C_2 & : \quad x = 2 \cos t, \quad y = -2 \sin t, \quad 0 \leq t \leq \pi .
\end{align*}
\]