Limit is a function. A function: For every input there can be only one output.

Idea is: As \( x \) approaches \( c \), is the function approaching a value?

\[ x \to c \quad \Rightarrow \quad f(x) \to L \]

Same value if approaching \( c \) from the left, right or any approach.

New Notation

\[
\lim_{x \to c} f(x) = L
\]

If the value \( L \) (must be a number) exist as \( x \to c \) we say that the limit exist at \( x = c \).

\[ \lim_{x \to c} f(x) = ? \]

A. If \( L \) exist then the limit exist, written as: \( \lim_{x \to c} f(x) = L \).

Two results

1. If \( \lim_{x \to c} f(x) = L = f(c) \) then \( f(x) \) is continuous at \( x = c \).
2. If \( \lim_{x \to c} f(x) = L \neq f(c) \) then \( f(x) \) is not continuous but there is a hole.
   
   If we want to make \( f(x) \) to be continuous at \( x = c \) then we set \( f(c) = L \).

B. If \( L \) does not exist it is written \( \lim_{x \to c} f(x) = DNE \)

Two results

1. There is a jump in the function. The function value on the left of \( c \) does not equal the function value to the right of \( c \). \( \lim_{x \to c^-} f(x) = L \neq M = \lim_{x \to c^+} f(x) \)

2. There is a vertical asymptote: The function is going to infinity \( f(x) \to \pm\infty \)

The book does allow this notation: \( \lim_{x \to c} f(x) = \pm\infty \)

\[ \lim_{x \to \infty} f(x) = L \]

As \( x \) gets extremely large is the function approaching a value?

Is there a horizontal asymptote?

Two results

1. If \( \lim_{x \to \infty} f(x) = L \) then there is a horizontal asymptote: \( y = L \).
2. If \( \lim_{x \to \infty} f(x) = \pm\infty \) then the limit does not exists.

\[ f(x) \to \infty \quad \text{the function is growing without bounds.} \]

\[ f(x) \to -\infty \quad \text{the function is decreasing without bounds.} \]

**Rule of 4 on Evaluating a limit function.** \( \lim_{x \to c} f(x) = L \)

Graphically: \( x \to c \) No matter how \( x \) approaches \( c \) the function seems to be approaching the same value. The function is approaching the same value on the right and left of \( c \). You must zoom in very closely if using your calculator.

\( x \to \infty \) \( L \) exist if there is a horizontal asymptote.

Numerically: Make a table of values by picking values of \( x \) and evaluating the function. At least 6 values so one can see the approach. You may need more if you can’t tell or rounding to a certain number of decimals. Remember \( x \neq c \).

\[ x \to c \]

Letting values of \( x \) be very close to \( c \) ie. \( \{ c + .001, c + .0001, c + .00001 \} \)

\[ x \to \infty \]

Pick values that are very large ie. 1 million, 1 billion, 1 trillion etc.

Algebraically: Use the limit rules. Usually you have to do some algebra first. See next page for rules.

Words: \( x \to \infty \) There is a horizontal asymptote. In story problems: “eventually” or “in the long run” ie. The flies will increase to a max number of flies of 560. \( \lim_{t \to \infty} f(t) = 560 \).

ie. The yam will reach oven temperature. The Ph balance will eventually stabilize to 4.
Properties of Limits use to evaluate a limit function. Assuming all the limits on the right hand side exist:

1. \( \lim_{x \to c} kf(x) = k \lim_{x \to c} f(x) \) where \( k \) is a real number.
2. \( \lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x) \)
3. \( \lim_{x \to c} (f(x)g(x)) = \left( \lim_{x \to c} f(x) \right) \left( \lim_{x \to c} g(x) \right) \)
4. \( \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \) \( \text{IF } \lim_{x \to c} g(x) \neq 0 \) or \( g(x) \not\to \infty \)

To use this rule the limit of the denominator must approach a non-zero number.

5. \( \lim_{x \to c} k = k \)
6. \( \lim_{x \to c} x = c \)
7. \( \lim_{x \to c} (f(x))^p = (\lim_{x \to c} f(x))^p \) Limits can move in and out of functions.
8. \( \lim_{x \to \infty} \frac{1}{x} = 0 \) and \( \lim_{x \to -\infty} \frac{1}{x} = 0 \) and \( \lim_{x \to \infty} \frac{1}{x^2} = 0 \)
9. \( \lim_{x \to \infty} e^{-x} = 0 \) and \( \lim_{x \to -\infty} e^x = 0 \)
10. If \( b > 1 \), then \( \lim_{x \to \infty} b^{-x} = 0 \) and \( \lim_{x \to -\infty} b^x = 0 \)
11. If \( 0 < b < 1 \) then \( \lim_{x \to \infty} b^x = 0 \) and \( \lim_{x \to -\infty} b^{-x} = 0 \)

Another note: \( \lim_{x \to c} \frac{N(x)}{D(x)} \) or \( \lim_{x \to \infty} \frac{N(x)}{D(x)} \) for these limits to possibly exist:

1. \( D(c) \neq 0 \) or \( D(c) \not\to \infty \) \[\text{Just use the limit rules and evaluate.}\]
2. If \( D(c) = 0 \) then \( N(c) = 0 \) \[\text{A hole}\]
3. If \( D \not\to \infty \)
   For type 2 and 3 one needs to do algebra before evaluating the limit. Must change the denominator to be approaching a non-zero value.

To make a function continuous when there is a hole in the graph. Find the limit of the function as \( x \) approaches the point of discontinuity. Set the function equal to this value at the point of discontinuity.

Example:

Evaluate \( \lim_{h \to 0} \frac{(3+h)^2 - 3^2}{h} = \lim_{h \to 0} \frac{(9+6h+h^2) - 9}{h} \)

\[= \lim_{h \to 0} \frac{6h+h^2}{h} = \lim_{h \to 0} (6+h) = \lim_{h \to 0}^{eval} 6+0 = 6 \]

Evaluate \( \lim_{t \to 5} \frac{t^2-9}{t-5} = DNE \) Since the denominator is approaching zero and the numerator is not. The conclusion at \( t = 5 \) must be Vertical Asymptotes.
Graphically

a. \( \lim_{x \to 1} g(x) = \)
b. \( \lim_{x \to -1} g(x) = \)
c. \( \lim_{x \to 2^-} g(x) = \)
d. \( \lim_{x \to 2^+} g(x) = \)
e. \( \lim_{x \to 2} g(x) = \)
f. \( \lim_{x \to 4^-} g(x) = \)
g. \( \lim_{x \to 4} g(x) = \)
h. \( g(1) = \)
i. \( g(2) = \)
j. \( g(4) = \)

Table  [For your homework you will be making a table of value – Use evaluate program.]

Use the following results to evaluate \( \lim_{x \to 3} T(x) = \)___________

<table>
<thead>
<tr>
<th>( x )</th>
<th>( T(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.99</td>
<td>7.4872</td>
</tr>
<tr>
<td>2.999</td>
<td>7.4905</td>
</tr>
<tr>
<td>2.9999</td>
<td>7.4985</td>
</tr>
<tr>
<td>2.99999</td>
<td>7.4999</td>
</tr>
<tr>
<td>3.00001</td>
<td>7.4992</td>
</tr>
<tr>
<td>3.0001</td>
<td>7.4989</td>
</tr>
<tr>
<td>3.001</td>
<td>7.4729</td>
</tr>
</tbody>
</table>

When you make a table you need enough decimal places so you can see it approaching a value. Need 4 or 5 decimals places in the table to approximate 3 decimal places.

Algebraically

Evaluate:

\[ \lim_{x \to -3} \frac{x^2 + 6x + 9}{x + 3} \]

\[ \lim_{x \to \infty} \frac{x^2 + 6x + 9}{3 + 2x^2} \]