Definition: The Jacobian matrix of a map \( f : N \rightarrow M \) is
\[
\begin{pmatrix}
\frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^n} \\
\vdots & & \vdots \\
\frac{\partial F^m}{\partial x^1} & \cdots & \frac{\partial F^m}{\partial x^n}
\end{pmatrix}
\] When this is a square matrix, its determinant is called the Jacobian determinant.

Definition: The rank of a smooth map \( F \) is the rank of the Jacobian matrix, which is the largest number of linearly independent columns of the matrix.

**Constant Rank Theorem:** Let \( N \) and \( M \) be manifolds of dimension \( n \) and \( m \) respectively. If \( f : N \rightarrow M \) has constant rank \( k \) is a neighborhood of a point \( p \in N \), then there are charts \((U, \phi)\) centered at \( p \) and \((V, \psi)\) centered at \( f(p) \) such that \( \psi \circ f \circ \phi^{-1}(r^1, ..., r^n) = (r^1, ..., r^k, 0, ..., 0) \)

**Constant-rank level set theorem:** If \( f : N \rightarrow M \) is smooth, and if \( f \) has constant rank \( k \) in a neighborhood of \( f^{-1}(c) \), then \( f^{-1}(c) \) is a regular submanifold of \( N \) of codimension \( k \).

**Proposition 1:** Let \( f : N \rightarrow M \) be a smooth function of manifolds where \( N \) and \( M \) have dimension \( n \) and \( m \) respectively.

1. \( f \) is an immersion at \( p \) iff the rank of the Jacobian matrix of \( F \) equals \( n \leq m \). \( f \) is an immersion if it is an immersion at all points.

2. \( f \) is a submersion at \( p \) iff the rank of the Jacobian matrix of \( F \) equals \( m \leq n \). \( f \) is a submersion if it is a submersion at all points.

**Immersion theorem:** If \( f : N \rightarrow M \) is an immersion at \( p \), then there are charts \((U, \phi)\) centered at \( p \) and \((V, \psi)\) centered at \( f(p) \) such that \( \psi \circ f \circ \phi^{-1}(r^1, ..., r^n) = (r^1, ..., r^n, 0, ..., 0) \) in a neighborhood of \( \phi(p) \).

**Submersion theorem:** If \( f : N \rightarrow M \) is a submersion at \( p \), then there are charts \((U, \phi)\) centered at \( p \) and \((V, \psi)\) centered at \( f(p) \) such that \( \psi \circ f \circ \phi^{-1}(r^1, ..., r^m, r^{m+1}, ..., r^n) = (r^1, ..., r^m) \) in a neighborhood of \( \phi(p) \).

**Inverse function theorem:** Let \( F : N \rightarrow M \) be a smooth map of manifolds with same dimension. Then \( F \) is locally invertible at \( p \in N \) iff its Jacobian determinant at \( p \) is nonzero.

**Implicit function theorem:** If \( F : U \rightarrow \mathbb{R}^m \) is smooth where \( U \) is an open subset of \( \mathbb{R}^n \times \mathbb{R}^m \), \( f(a, b) = 0 \), and the Jacobian determinant of \( f \) at \( (a, b) \) is not zero, then there is a smooth function \( h : A \rightarrow B \) such that \( f(x, y) = 0 \iff y = h(x) \) in \( A \times B \) where \( A \times B \) is an open subset of \( U \) containing \( (a, b) \).

**Whitney’s theorem:** Any smooth \( n \)-dimensional manifold can be smoothly embedded into \( \mathbb{R}^{2n} \).

**Stoke’s Theorem:** Let \( \omega \) be a smooth \((n-1)\)-form with compact support on an oriented \( n \)-dimensional manifold \( M \). Then \( \int_M d\omega = \int_{\partial M} \omega \).
Proposition 2:  
1. $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$
2. If $\omega = \sum_I a_I dx^I$, then $d\omega = \sum_I da_I \wedge dx^I$
3. $d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^{\text{deg} \omega} \omega \wedge d\tau$

Proposition 3: Let $\omega$ be an $n$-form on $S^n$. $\int_{S^n} \omega = 0 \iff \omega$ is exact

Proof: If $\omega$ is exact, then $\omega = d\tau$. So $\int_{S^n} \omega = \int_{S^n} d\tau = \int_{\partial S^n} \tau = \int_{\emptyset} \tau = 0$. For the converse, note that $H^n(S^n) = \mathbb{R}$ and that the volume $\nu$ form on $S^n$ does not integrate to zero. Therefore, the cohomology classes of the $n$-forms are $[r\nu]$ where $r \in \mathbb{R}$. So every $n$-form $\omega$ of on $S^n$ can be written in the form $\omega = r\nu + \tau$ where $\tau$ is exact. $\int_{S^n} \omega = r\int_{S^n} \nu$. If $\int_{S^n} \omega = 0$, then $r = 0$. So $\omega = \tau$ is exact. QED

Proposition 4: A form $\omega = \sum_I a_I dx^I$ is smooth iff all the $a_I$ are smooth functions on each chart.

Proposition 5:
1. A manifold is orientable iff it has a nowhere-vanishing smooth top form.
2. A manifold has a trivial tangent bundle iff it has a smooth global frame.
3. If a manifold has a smooth global frame, then it is orientable.

Definition: A smooth partition of unity on a manifold $M$ is a collection of nonegative function such that their sum is 1 and every point in $M$ has a neighborhood that intersects only finitely many of the supports of the functions.

Proposition 6:
1. Every manifold has a smooth partition of unity with each function having compact support and they may be chosen so that each support lies inside a set of an open cover.
2. Every manifold has a smooth partition of unity that is subordinate to any chosen open cover.

Proposition 7:
1. For a function $f : V \to W$ between vector spaces the pullback of the differential $n$-form $\omega$ is $f^*(\omega)(v_1, \ldots, v_n) = \omega(f_*v_1, \ldots, f_*v_n)$
2. The pullback is linear $f^*(a\omega + b\tau) = af^*(\omega) + bf^*(\tau)$
3. The pullback commutes with the differential $f^*d\omega = df^*\omega$
4. The pullback distributes with the wedge produce $f^*(\omega \wedge \tau) = f^*(\omega) \wedge f^*(\tau)$
Definition: The push forward of a map \( f : M \to N \) is defined by \( f_*(\frac{\partial}{\partial x^j}) = \sum_i \frac{\partial f^i}{\partial x^j} \frac{\partial}{\partial y^i} \).

**Hairy ball theorem:** If \( n \) is even, then any continuous tangent vector field on \( S^n \) must vanish.

**Proposition 8:** \( S^0, S^1, S^3, S^7 \) only spheres with trivial tangent bundle.

**Borsuk-ulam theorem:** For every continuous map \( f : S^2 \to \mathbb{R}^2 \) there is a pair of antipodal points \( x \) and \(-x\) in \( S^2 \) such that \( f(x) = f(-x) \).

Definition: The degree of a map \( f : S^n \to S^n \) is the integer \( d \) such that \( f_*(\alpha) = d\alpha \), where \( f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n) \).

**Proposition 9:**

1. \( \deg(f) = 0 \) if \( f \) is not surjective.
2. \( \deg(f \circ g) = \deg(f) \cdot \deg(g) \)
3. \( \deg(f) = \pm 1 \) if \( f \) is a homotopy equivalence
4. \( \deg(f) = (-1)^{n+1} \) if \( f \) is the antipodal map
5. \( \deg(f) = \deg(g) \) if \( f \simeq g \). The converse is also true for \( n > 0 \).

Definition: The connected sum \( M \# N \) of two surfaces is the space constructed by removing a disc from each and identifying the boundary circles of the removed discs.

**Van Kampen’s theorem:** If \( X \) is the union of path-connected open sets \( A_\alpha \) each containing the same base point, and each triple intersection of the \( A_\alpha \) is path-connected, then \( \Pi_1(X) \cong (\ast_\alpha \Pi_1(A_\alpha))/N \) where \( N \) is the normal subgroup generated by all elements of the form \( i_{\alpha\beta}(\omega)i_{\beta\alpha}^{-1}(\omega) \). \( i_{\alpha\beta} \) maps into \( \Pi_1(A_\alpha) \) and \( i_{\beta\alpha} \) maps into \( \Pi_1(A_\beta) \).

**Proposition 10:** If \( X \) is path-connected, \( H_1(X) = \Pi_1(X)^{ab} \)

**Mayer-Vietoris:** Let \( U, V \) be open sets whose union is the entire space \( X \). Then the Mayer-Vietoris sequences are exact.

1. For homology \( H_n \), you go from \( U \cap V \) to \( U \cup V \) to \( X = U \cup V \) and then to a lower homology, decreasing \( n \)
2. For cohomology \( H^n \), you go from \( X = U \cup V \) to \( U \cap V \) to \( U \cap V \) and to a higher cohomology, increasing
3. For compact support cohomology \( H^n_c \), go up as in cohomology but reverse the direction. \( n \)
Proposition 11: The alternating sum of the degrees in the Mayer-Vietoris sequence is zero

Proof: This follows from exactness. QED

Proposition 12: $H^n(M \times \mathbb{R}) \cong H^n(M)$

Kunneth formula: For manifolds $M$ and $F$, $H^n(M \times F) = \oplus_{p+q=n} H^p(M) \otimes \mathbb{R} H^q(F)$

Proposition 13: $\tilde{H}_n(\vee \alpha X_\alpha) \cong \oplus_{\alpha} \tilde{H}_n(X_\alpha)$ provided that $(X_\alpha, x_\alpha)$ are good pairs, where $x_\alpha$ are the base points.

Proposition 14: $H_0(x) \cong \mathbb{Z}^k$, where $k$ is the number of path-components.

Definition: The reduced homology $\tilde{H}_n$ is simply $H_n$ for $n \neq 0$ and has one less $\mathbb{Z}$ summand for $n = 0$.

Definition: A singular $n$-simplex is a continuous map $\sigma : \Delta^n \to X$. $\partial \sigma = \sum_{i} (-1)^i \sigma|\text{face}_i$. An $n$-chain is a formal finite $\mathbb{Z}$-linear combination of $n$-simplices. The singular homology is those chains without boundary mod those that are the boundary of other chains.

Singular homology:

Definition: The cellular homology of a space $X$ is found by considering the sequence $\cdots \to H_{n+1}(X^{n+1}, X^n) \to H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots$, where $H_n(X^n, X^{n-1}) \cong \mathbb{Z}^k$ where $k$ is the number of $n$-cells in the complex. The maps between the $H_n$ are the boundary maps, and the cellular homology $H_n(X) = \ker d_n / \text{im } d_{n-1}$ where $d_n$ maps from $H_n(X^n, X^{n-1})$. That is, it maps from the $n$-cells

Homology and Cohomology of common spaces.

1. $S^n$ The de Rham cohomology is $\mathbb{R}$ in dimension 0 and $n$ and is zero otherwise. For homology replace $\mathbb{R}$ by $\mathbb{Z}$.

2. $\mathbb{R}^n$ The de Rham cohomology is $\mathbb{R}$ in dimension 0 and is zero otherwise (Poincare lemma). For homology replace $\mathbb{R}$ by $\mathbb{Z}$.

3. $T^2$. The homology is $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}$ for $n = 0, 1, 2$ respectively, otherwise it’s zero.

Definition: Two maps $f_0 : X \to Y$ and $f_1 : X \to Y$ are homotopic if there exists a continuous function $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. For two paths to be homotopic the end points must be fixed.

Definition: $X \simeq Y := X$ is homotopy equivalent to $Y$. This means there exist map $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity.
Definition: $G \cong H := G$ and $H$ are isomorphic

**Proposition 15:** Homotopy equivalent spaces have the same homology and cohomology.

Definition: A covering space of a space $X$ is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ for which there exists an open cover $\{U_{\alpha}\}$ of $X$ such that $p^{-1}(U_{\alpha})$ is a disjoint union of open sets in $\tilde{X}$ each of which is homeomorphic to $U_{\alpha}$ via $p$. Note $p$ need not be surjective.

Definition: Two covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are isomorphic if there exists a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ such that $p_1 = p_2 \circ f$.

Definition: The deck transformations $G(\tilde{X})$ of a covering space is the group of isomorphisms of the covering space with itself.

Definition: A covering space is normal if for each pair of lifts $\tilde{x}, \tilde{x}'$ of $x$ there is a deck transformation sending one to the other.

**Prop 1.33 in Hatcher:** Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space and $Y$ be a path-connected and locally path-connected space. Then a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ of $f : (Y, y_0) \to (X, x_0)$ exists iff $f_*(\Pi_1(Y, y_0)) \subseteq p_*(\Pi_1(\tilde{X}, \tilde{x}_0))$. The lift is unique once a based point is fixed.

**Proposition 16:** If $X$ is path-connected, locally path-connected, and semi-locally simply-connected, then there is a one to one correspondence between subgroups $H \leq \Pi_1(x)$ and covering spaces $X_H$ of $X$. $p_*(\Pi_1(X_H)) = H$.

**Proposition 17:** Let $p : \tilde{X} \to X$ be a path-connected covering space of the path-connected locally path-connected space $X$. Let $H = p_*(\Pi_1(\tilde{X}))$. Then

1. The covering space is normal iff $H$ is a normal subgroup of $\Pi_1(X)$
2. $G(\tilde{X}) \cong N(H)/H$, where $N(H)$ is the normalizer of $H$ in $\Pi_1(X)$
3. It follows that for the universal cover, $G(\tilde{X}) \cong \Pi_1(X)$

Definition: The action of a group $G$ on a space $Y$ is properly discontinuous if each $y \in Y$ has a neighborhood $U$ such that the $g(U)$ are disjoint for different $g \in G$.

**Prop 1.40 in hatcher** If an action of a group $G$ on a path-connected and locally path-connected space $Y$ is properly discontinuous, then $G \cong \Pi_1(Y/G)/p_*(\Pi_1(Y))$

**Proposition 18:** If $G \leq Y$ is a finite group acting on $Y$ via multiplication and multiplication on
Y is continuous and every element in Y has an inverse, then the action is properly discontinuous.

Proof: Let y ∈ Y. Then the gy are all distinct, because every element in Y has an inverse. So for each g ∈ G, there is an open neighborhood V_{gy} such that V_{gy} and V_{g'y} are all disjoint. This is possible because G is finite. Let U_g = g^{-1}(V_g), and let U = Π_{g ∈ G} U_g. Each U_g contains y because gy ∈ V_g. So U contains y, and it is open, because it is a finite intersection of open sets. Finally, note that g(U) ⊆ g(U_g) ⊆ V_g. So that the different g(U) are disjoint. Therefore, the action is properly discontinuous. QED

Proposition 19: The two dimensional fractal tree is the universal cover of S^1 ∨ S^1. The three dimensional fractal tree is the universal cover of S^1 ∨ S^1 ∨ S^1.

Definition: For a finite CW complex, the Euler characteristic χ(X) = ∑_{n} (-1)^n c_n where c_n is the number of n-cells in X

Proposition 20: χ(X) = ∑_{n} (-1)^n rankH_n(X). Hence, χ(X) depends only on homotopy type, and is independent of the CW structure on X

Cauchy Riemann equations: If f(x + iy) = u(x, y) + iv(x, y) is differentiable and u, v are real, then \frac{∂u}{∂x} = \frac{∂v}{∂y} and \frac{∂u}{∂y} = -\frac{∂v}{∂x}.

Picardi’s great theorem: If f has an essential singularity at z_0 ∈ U where U is a neighborhood of z_0, then f(U) = C or C \ {point}.

Residue theorem: If f is meromorphic on D and γ is a simple closed curve not passing through a pole of f, then \int_{γ} f(z) dz = 2πi ∑_{poles z_0 enclosed by γ} Res_{z_0} f, where Res_{z_0} f(z) is the coefficient of the term \frac{1}{z-z_0} in the Laurent series.

Cauchy integral formula: If f is analytic, f(z) = \frac{1}{2\pi i} \int_{γ} \frac{f(s)}{s-z} ds, where γ is a path around z.

Open mapping theorem: Any non-constant analytic function is an open mapping.

Maximum modulus principle: If f(z) is a nonconstant function on an open set U, f does not attain a maximum modulus, |f(z)|.

Louiville’s theorem: Any bounded entire function is constant.

Picardi’s little theorem: If f is entire and f(C) omits at least two values, then f is constant.

Riemann’s mapping theorem: If U ⊆ C is open and simply connected and U ≠ C, then there exists an analytic isomorphism U → D, where D is the unit disk.
Conformal maps are all diffeomorphisms.

conformal mapping ingredients: Translation $z \mapsto z + z_0$, Rotation $z \mapsto e^{i\theta}z$, Wrap around origin $z \mapsto z^k$ ($k$ not necessarily an integer), sin($z$) half strip to upper half plane, log($z$) upper half of unit disc to half horizontal strip in second quadrant, $z \mapsto (z - i)/(z + i)$ takes upper half plane to disc, exponential takes horizontal strip to upper half plane.

Example: quarter disc in first quadrant to vertical half strip in first quadrant.

1. $z \mapsto z^2$ Quarter to half disk

2. $z \mapsto \log z$ half disk to half horizontal strip in second quadrant

3. $z \mapsto e^{-i\pi/2}z$ rotates clockwise 90 degrees.